14. Taylor expansion (28/10/2021)

Goals.

(1) Higher-order approximation
(2) Combining expansions

Last Time. MVT: $\frac{f(x)-f(a)}{x-a} = f'(c)$ for some c between a, x.

average rate of change

instantaneous rate of change

$\Leftrightarrow \quad f(x) = f(a) + f'(c)(x-a)$

Linear approximation: $f(x) \approx f(a) + f'(a)(x-a)$

Worksheet (1)

Worksheet (2)

Midterm material
Weeks 1–6
on
course syllabus

If $T_n(x) = c_0 + c_1 x + \dots + c_n x^n$ (expanding about 0)

k'th derivative of $c_k x^k$ is $(1 \cdot 2 \cdot 3 \cdots k) \cdot c_k$

so to match k'th derivative of f use

$$c_k = \frac{f^{(k)}(a)}{k!} \qquad k! = 1 \cdot 2 \cdot 3 \cdots k$$

1. Taylor approximation

(1) (Review) Use linear approximations to estimate:

(a) $\log\frac{4}{3}$ and $\log\frac{2}{3}$. Combine the two for an estimate of $\log 2$.

take $f(x)=\log x$, $a=1$. Then $f'(x)=\frac{1}{x}$

so $f(1)=\log 1=0$, $f'(1)=\frac{1}{1}=1$

so $f(\frac{4}{3})\approx 0+1(\frac{4}{3}-1)=\frac{1}{3}$

$f(\frac{2}{3})\approx 0+1(\frac{2}{3}-1)=-\frac{1}{3}$

$\frac{4/3}{2/3}=2$ $(2\cdot\frac{2}{3}=4/3$

so $\log 2+\log 2/3=\log 4/3$

so $\log 2\approx\frac{1}{3}-(-\frac{1}{3})=2/3$

(b) $\sin 0.1$ and $\cos 0.1$.

Expand about $a=0$. Tangent line to $y=\sin x$ at $x=0$ is $y=x$

$(\sin x)'=\cos x$, $\cos 0=1$ Tangent line to $y=\cos x$ at $x=0$ is $y=1$

so to 1st order, $\sin 0.1\approx 0.1$

$\cos 0.1\approx 1$.

Date: 28/10/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$f(x)=\sin x$, $a=0$: $f(0)=\sin 0=0$; $f(x)\approx f(0)+f'(0)(x-0)$

$f'(0)=\cos 0=1$ $=0+1\cdot x$

(2) Let $f(x) = e^x$

(a) Find $f(0), f'(0), f^{(2)}(0), \cdots$

(b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.

(c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.

(d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.

(e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

(a) $f'(x) = e^x$, $f^{(2)}(x) = e^x$, $f^{(3)}(x) = e^x$,

so $f(0) = e^0 = 1$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = 1$, $f^{(3)}(0) = 1$, ...

(b) $T_0(x) = 1$ is the simplest polynomial s.t. $T_0(0) = 1$

(c) $T_1(x) = 1 + x$ is the line tangent to $y = f(x)$ at $a = 0$

(if not sure, try $T_1 = 1 + ax$ $T_1' = a$, set $a = 1$ to get derivative 1)

(d) try $T_2(x) = 1 + x + c_2 x^2$

$T_2''(x) = 0 + 2c_2$ want $T_2''(0) = 1$ so choose $c_2 = \frac{1}{2}$.

(e) try $T_3(x) = 1 + x + \frac{1}{2}x^2 + c_3 x^3$, $T_3'''(x) = 0 + 6c_3$

to get $T^{(3)}(0) = 1$ choose $c_3 = \frac{1}{6} = \frac{1}{1\cdot2\cdot3}$

get $T_3(x) = 1 + \frac{x}{1} + \frac{x^2}{1\cdot2} + \frac{x^3}{1\cdot2\cdot3}$.

(3) Do the same with $f(x) = \ln x$ about $x = 1$.

$f'(x) = \frac{1}{x}$ $\quad f''(x) = -\frac{1}{x^2}$ $\quad f'''(x) = \frac{2}{x^3}$

$f(1) = 0$ $\quad f'(1) = 1$ $\quad f''(1) = -1$ $\quad f^{(3)}(1) = 2$

take $T_0(x) = 0$; $\quad T_1(x) = x - 1$

try $T_2(x) = (x-1) + C_2(x-1)^2$

$T_2''(x) = 0 + 2C_2$ so choose $C_2 = -\frac{1}{2} = \frac{1}{2} \cdot f^{(2)}(1)$

set $T_2(x) = (x-1) - \frac{1}{2}(x-1)^2$

similarly $\quad C_3 = \frac{1}{6} f^{(3)}(1) = \frac{1}{3}$

so $T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(4) Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)

Let $f(x) = \frac{1}{1-x}$, $f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$, $f^{(3)}(x) = \frac{6}{(1-x)^4}$

$f^{(4)}(x) = \frac{24}{(1-x)^5}$, so $f(0) = 1$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = 2$, $f^{(3)}(0) = 6$, $f^{(4)}(0) = 24$.

So $T_4(x) = 1 + \frac{1}{\underset{\leftarrow 1!}{1}}(x) + \frac{2}{\underset{\nwarrow 2!}{2}}x^2 + \frac{6}{\underset{\nwarrow 3!}{6}}x^3 + \frac{24}{\underset{\nwarrow 4!}{24}}x^4 = 1 + x + x^2 + x^3 + x^4$

(plug in $x = -y$ to get:

$$\frac{1}{1+y} \approx 1 - y + y^2 - y^3 + y^4 + \ldots$$)

(5) Find the nth order expansion of $\cos x$, and approximate $\cos 0.1$ using a 3rd order expansion

let $f(x) = \cos x$; $f'(x) = -\sin x$; $f^{(2)}(x) = -\cos x$; $f^{(3)}(x) = \sin x$

$f^{(4)}(x) = \cos x$ $f^{(5)}(x) = -\sin x$; repeat

$f(0) = 1$ $f^{(1)}(0) = 0$ $f^{(2)}(0) = -1$ $f^{(3)}(0) = 0$

$f^{(4)}(0) = 1$ $f^{(5)}(0) = 0$, -1, 0,

repeat

so Taylor/MacLaurin expansion of $\cos x$ is

$$1 - \underset{\substack{\uparrow \\ 2!}}{\frac{1}{2}} x^2 + \underset{\substack{\uparrow \\ 4!}}{\frac{1}{24}} x^4 - \underset{\substack{\uparrow \\ 6!}}{\frac{1}{720}} x^6 + \underset{\substack{\uparrow \\ 8!}}{\frac{1}{40,320}} x^8 - \cdots$$

$\cos 0.1 \simeq 1 - \frac{1}{200}$

$(2n)$th term is $\frac{(-1)^n}{(2n)!} x^{2n}$.

& $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

(6) (Final, 2015) Let $T_3(x) = 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?

Here $C_2 = 12 = \dfrac{f''(3)}{2}$ so $f''(3) = 24$

2. New from old

(7) (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x=0$.

Extra idea: ① approximate $\sin x \approx x - \frac{1}{6}x^3$

$x+1 \approx 1 + x + 0x^2 + 0x^3$

② mutiply:

$$(1+x)\left(x - \frac{x^3}{6}\right) = x + x^2 - \frac{x^3}{6} - \frac{x^4}{6}$$

so to $\underline{3^{rd}\text{ order}}$ $(1+x)\sin x = x + x^2 - \frac{x^3}{6}$