13. The Mean Value Theorem (26/10/2021)

Goals.

(1) The Mean Value Theorem ← from f' to f

(2) The Linear approximation

Last Time. **Exponential growth/decay**

$N(t) = C \cdot e^{kt}$ $\left(\begin{matrix} k>0 : \text{growth} \\ k<0 : \text{decay} \end{matrix}\right)$

Don't have to use base e. Eg. Can write $N_0 \cdot \left(\frac{1}{2}\right)^{t/\tau}$

(τ = "half-life"), $2^{-t/\tau} = \left(\frac{1}{2}\right)^{t/\tau} = e^{-\frac{\log 2}{\tau} t}$

Example: NLC.

(often arises from models $y' = ky$)

After driving 1 hr I covered 60 Km

Average velocity $\frac{60 \text{ km}}{h}$.

Could I have always been driving faster than $60 \frac{\text{km}}{h}$?

" " " " " " slower " " ?

$\Rightarrow$ (if life is nice) at some point I was going $60 \frac{\text{km}}{h}$ exactly.

Math 100 – WORKSHEET 17
THE MEAN VALUE THEOREM; LINEAR APPROXIMATION

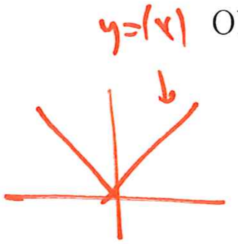
1. Average slope vs Instantenous slope

(1) Let $f(x) = e^x$ on the interval $[0, 1]$. Find all values of c so that $f'(c) = \frac{f(1)-f(0)}{1-0}$.

need c s.t.

$$e^c = \frac{e-1}{1-0} = e-1 \quad \text{so} \quad c = \log(e-1)$$

(2) Let $f(x) = |x|$ on the interval $[-1, 2]$. Find all values of c so that $f'(c) = \frac{f(2)-f(-1)}{2-(-1)}$



$$f'(c) = \begin{cases} -1 & c<0 \\ +1 & c>0 \end{cases} \quad \text{while} \quad \frac{f(2)-f(-1)}{2-(-1)} = \frac{1}{3}.$$

Date: 26/10/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

The Mean Value thm

~~[illegible]~~ **Thm:** let f be differentiable on $[a,b]$.
Then there is $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Takeaways:

(1) if can solve for c, no need for MVT

(2) really need to check for differentiability.

Example: Suppose $f'(x) > 0$ on (a,b)

take f' info

then $\frac{f(b)-f(a)}{b-a} = f'(c) > 0$

invoke MVT

endgame with $f(a)$, $f(b)$

so $f(b)-f(a) > 0$ so $f(b) > f(a)$

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2. The Mean Value Theorem

(3) Show that $f(x) = 3x^3 + 2x - 1 + \sin x$ has exactly one real zero. (Hint: let a, b be zeroes of f. The MVT will find c such that $f'(c) = ?$)

Suppose f had to zeroes $a \neq b$. f is defined by formula everywhere hence everywhere diff. By MVT there is c between a, b s.t

$$f'(c) = \frac{f(b) - f(a)}{b-a} = \frac{0-0}{b-a} = 0$$

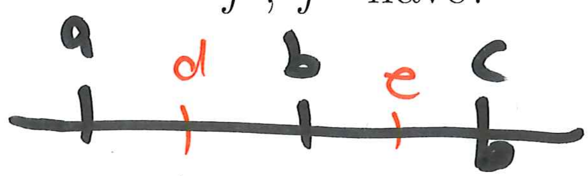
But $f'(x) = 9x^2 + 2 + \cos x \geq 0 + 2 - 1 = 1 > 0$

so such c does not exist, so a, b do not exist: f has at most one zero

(to show at least one use IVT)

(4) (Final, 2015)

(a) Suppose f, f', f'' are all continuous. Suppose f has at least three zeroes. How many zeroes must f', f'' have?



a, b, c zeroes of f

For any f, if $f(a) = f(b) = 0$ & f' exists, by MVT get point x between a, b s.t. $f'(x) = \frac{f(a) - f(b)}{a - b} = 0$

Applying to f with intervals $[a,b]$, $[b,c]$ get at least two zeroes of f', say $d \in (a,b)$, $e \in (b,c)$. Applying to f' on $[d,e]$ get at least one zero $g \in (d,e)$ of f''.

(b) [Show that $2x^2 - 3 + \sin x + \cos x = 0$ has at least two solutions] ← Use IVT

(c) Show that the equation has at most two solutions.

Suppose not. Then $f(x) = 2x^2 - 3 + \sin x + \cos x$ would have at least three. By part (a), f' would have two roots and f'' would have at least one

But $f''(x) = 4 - \sin x - \cos x \geq 4 - 1 - 1 = 2 > 0$

so f'' never vanishes, so f can have at most two roots

(5) (Final, 2012) Suppose $f(1) = 3$ and $-3 \leq f'(x) \leq 2$ for $x \in [1, 4]$. What can you say about $f(4)$?

f' exists on $[1,4]$ so by MVT $\dfrac{f(4)-f(1)}{4-1} = f'(c)$ for some $1 < c < 4$.

so $-3 \leq \dfrac{f(4)-f(1)}{4-1} \leq 2$

$$-6 = \underbrace{3}_{f(1)} + \underbrace{(-3)}_{f' \text{ info}} \underbrace{(4-1)}_{\Delta x} \leq f(4) \leq \underbrace{3}_{f(1)} + \underbrace{2}_{f' \text{ info}} \cdot \underbrace{(4-1)}_{\text{length of interval } \Delta x} = 9$$

(6) Show that $|\sin a - \sin b| \leq |a - b|$ for all a, b.

(7) Let $x > 0$. Show that $e^x > 1 + x$ and that $\log(1 + x) < x$.

The linear approximation

Recap: MVT says: given points a, x have c between them s.t.

$$f(x) = f(a) + f'(c)(x-a)$$

useful for proofs: importing f' info into $f(x)$

bad for computation:

Instead guess (hope) that $f'(c)$ not too far from $f'(a)$; approximate

tangent line

$$f(x) \approx f(a) + f'(a)(x-a)$$

(choose a where we can compute $f(a)$, $f'(a)$)

3. The Linear Approximation

(8) Use a linear approximation to estimate

(a) $\sqrt{1.2}$

Let $f(x) = \sqrt{x}$, try $a=1$ (extra credit: $a=1.21$)

$f'(x) = \frac{1}{2\sqrt{x}}$, so $f(1)=1$, $f'(1)=\frac{1}{2}$, $f(1.2) \approx 1 + \frac{1}{2}(0.2) = 1.1$

displacement $1.2-1$

using $a=1.21$

$f(a) = 1.1$, $f'(a) = \frac{1}{2.2}$, $f(1.2) \approx 1.1 + \frac{1}{2.2}\left(-\frac{1}{100}\right)$

$= 1.1 - \frac{1}{2200}$.

(b) (Final, 2015) $\sqrt{8}$

Again $f(x) = \sqrt{x}$ at $x=8$, use linear approximation about $a=9$

Get $f(8) \approx f(9) + f'(9)(8-9) = 3 + \frac{1}{6}(-1) = 2\frac{5}{6}$.

(c) (Final, 2016) $(26)^{1/3}$

Use $f(x) = x^{1/3} = \sqrt[3]{x}$, expand about $a = 27$

$f'(x) = \frac{1}{3} x^{-2/3}$ $\quad f'(27) = \frac{1}{3} \cdot (27)^{-2/3} = \frac{1}{27}$.

(d) log 1.07

Use $f(x) = \log x$, $a = 1$: $\log 1 = 0$

$f'(x) = \frac{1}{x}$, $f'(1) = 1$.

Ors $f(x) = \log(1+x)$ expanded about $a = 0$

(now $\log(1+x) \approx \log 1 + \frac{1}{1} \cdot x = x$