6. The Derivative (28/9/2021)

Goals.

(1) The derivative at a point
(2) The derivative as a function
(3) Power laws and polynomials

Last Time.

__IVT__: Find root of $g(x)=f(x)$ by setting $h(x)=f(x)-g(x)$, find a,b s.t. $h(a), h(b)$ have different signs, check that h is cts. Get $a<c<b$ s.t. $h(c)=0 \iff g(c)=f(c)$

__Def__: Let f be defined near a.
The derivative of f at a is

$$f'(a)=\frac{df}{dx}(a)=\left.\frac{df}{dx}\right|_{x=a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

__if the limit exists.__

(if $f'(a)$ exists we say f is __differentiable__ at a)

__verb:__ __differentiate__

Math 100 – WORKSHEET 6
THE DERIVATIVE

1. Definition of the derivative

Definition. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

(1) Find $f'(a)$ if

(a) $f(x) = x^2$, $a = 3$.

Here $f(a+h) = f(3+h) = (3+h)^2$

$f(a) = f(3) = 3^2 = 9$

so $\frac{f(a+h)-f(a)}{h} = \frac{(3+h)^2 - 9}{h} = \frac{6h+h^2}{h} = 6+h \xrightarrow[h\to 0]{} 6$

(b) $f(x) = \frac{1}{x}$, any a.

$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0} \frac{\frac{a-(a+h)}{a(a+h)}}{h}$

$= \lim_{h\to 0} \frac{-h}{h(a+h)a} = \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}$

Date: 28/9/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(c) $f(x) = x^3 - 2x$, any a (you may use $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$).

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(2) Express the limits as derivatives: $\lim_{h\to 0} \frac{\cos(5+h) - \cos 5}{h}$, $\lim_{\cancel{h}\to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin x - \sin 0}{x - 0}$

$f(x) = \sin x$

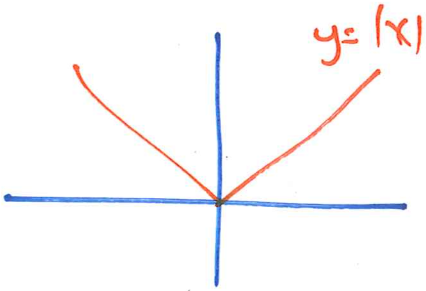
$a = 0$

$f(x) = \cos x$

$a = 5$

Fact: If f is diff. at a, then f is cts at a.

But



is cts, but not diff at 0

If $f'(x)$ exists for x in an interval, can think of the <u>derivative function</u> f'.

$$\left(f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right)$$

Differentiation rules

<u>Example:</u> $(x^n)' = n x^{n-1}$ (for $n \in \mathbb{R}$)

power law

(basically checked $n=2$ earlier)

$$(-x+x^2)' = -1 + 2x$$

$$(af + bg)' = af' + bg'$$

Notation: $f' = \frac{df}{dx} = Df = D_x f$, $\frac{dy}{dx}$ if $y = f(x)$ all valid since $\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{f(x)}{x}$, $\lim_{x\to 0} \frac{f(x)}{x}$ exists, i.e. f is diff. at 0.

(3) (Final, 2015) Is the function

$$f(x) = \begin{cases} \sqrt{1+x^2} - 1 & x \le 0 \\ x^2 \cos\frac{1}{x} & x > 0 \end{cases}$$

differentiable at $x = 0$?

$f(0) = \sqrt{1+0} - 1 = 0$

Need to compute $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$

def'n of $f'(0)$

$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{\sqrt{1+x^2}-1}{x}$

$= \lim_{x\to 0^-} \frac{(1+x^2)-1}{x} \cdot \frac{1}{\sqrt{1+x^2}+1} = \lim_{x\to 0^-} \frac{x}{\sqrt{1+x^2}+1} = 0$

$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{x^2\cos(\frac{1}{x})}{x}$

$= \lim_{x\to 0^+} x\cos(\frac{1}{x}) = 0$ squeeze thm

$-1 \le \cos\frac{1}{x} \le 1 \overset{x>0}{\Rightarrow} -x \le x\cos(\frac{1}{x}) \le x$, $\underset{x\to 0}{-x \to 0}$, $\underset{x\to 0}{x \to 0}$

2. Linear combinations; power laws

(4) Let $g(y) = Ay^{5/2} + y^2$. Suppose that $g'(4) = 0$. What is A?

$\frac{dg}{dy} = \frac{d(Ay^{5/2})}{dy} + \frac{d(y^2)}{dy} = A\cdot\frac{5}{2}\cdot y^{5/2-1} + 2y^{2-1} =$

$\frac{5A}{2} y^{3/2} + 2y$ so $g'(4) = \frac{5A}{2}\cdot 8 + 2\cdot 4 = 20A + 8$

so $20A + 8 = 0$ so $A = -\frac{8}{20} = -2/5$.

(5) Find the *second* derivative of $5t + 3\sqrt{t}$

If $y = 5t + 3\sqrt{t}$, $\frac{dy}{dt} = 5 + \frac{3}{2}t^{-1/2}$

so $y'' = \frac{d^2y}{dt^2} = 0 + \frac{3}{2}\cdot(-\frac{1}{2})t^{-3/2} = -\frac{3}{4}t^{-3/2}$

Q: Say $f(x) = \begin{cases} \sqrt{1+x^2} - 1 & x \leq 0 \\ x^2 \cos \frac{1}{x} & x > 0 \end{cases}$

isn't $f'(0)$ given by $\lim\limits_{x \to 0^-} \frac{f(x)}{x}$?

$$f(x) = \begin{cases} \sqrt{1+x^2} - 1 & x < 0 \\ 0 & x = 0 \\ x^2 \cos \frac{1}{x} & x > 0 \end{cases}$$