Lior Silberman's Math 412: Problem Set 2 (due 24/9/2019)

Practice

P1 Let $\{V_i\}_{i\in I}$ be a family of vector spaces, and let $A_i \in \mathrm{End}(V_i) = \mathrm{Hom}(V_i, V_i)$.
  (a) Show that there is a unique element $\bigoplus_{i\in I} A_i \in \mathrm{End}(\bigoplus_{i\in I} V_i)$ whose restriction to the image of V_i in the sum is A_i.
  (b) Carefully show that the matrix of $\bigoplus_{i\in I} A_i$ in an appropriate basis is block-diagonal.

P2 Construct a vector space W and three subspaces U, V_1, V_2 such that $W = U \oplus V_1 = U \oplus V_2$ (internal direct sums) but $V_1 \neq V_2$.

Direct sums

1. Give an example of $V_1, V_2, V_3 \subset W$ where $V_i \cap V_j = \{\underline{0}\}$ for every $i \neq j$ yet the sum $V_1 + V_2 + V_3$ is not direct.

2. (Diagonability)
   DEF A square matrix $A \in M_n(F)$ is diagonable (over F) if there exists an invertible matrix $S \in \mathrm{GL}_n(F)$ such that SMS^{-1} is diagonal.
   (a) Show that $A \in M_n(F)$ is diagonabke iff there exist n one-dimensional subspaces $V_i \subset F^n$ such $F^n = \bigoplus_{i=1}^n V_i$ and $A(V_i) \subset V_i$ for all i.
   (b) Let $T \in \mathrm{End}_F(V)$. For each $\lambda \in F$ let $V_\lambda = \mathrm{Ker}(T - \lambda)$ be the corresponding eigenspace. Let $\mathrm{Spec}_F(T) = \{\lambda \in F \mid V_\lambda \neq \{0\}\}$ be the set of eigenvalues of T. Show that the sum $\sum_{\lambda\in\mathrm{Spec}_F(T)} V_\lambda$ is direct.
   (c) Call $T \in \mathrm{End}_F(V)$ *diagonable* if its matrix with respect to some basis is diagonable. Show that T is diagonable iff $\sum_{\lambda\in\mathrm{Spec}_F(T)} V_\lambda = V$.

3*. Let $\{V_i\}_{i=1}^r$ be subspaces of W with $\sum_{i=1}^r \dim(V_i) > (r-1)\dim W$. Show that $\bigcap_{i=1}^r V_i \neq \{\underline{0}\}$.

Quotients

4. Write $M_n(F)$ for the space of $n \times n$ matrices with entries in F. Let $\mathfrak{sl}_n(F) = \{A \in M_n(F) \mid \mathrm{Tr} A = 0\}$ and let $\mathfrak{pgl}_n(F) = M_n(F)/F \cdot I_n$ (matrices modulu scalar matrices). Suppose that n is invertible in F (equivalently, that the characteristic of F does not divide n). Show that the quotient map $M_n(F) \to \mathfrak{pgl}_n(F)$ restricts to an isomorphism $\mathfrak{sl}_n(F) \to \mathfrak{pgl}_n(F)$.

5. (Quotients and complements) Let W be a vector space and let $U \subset W$ be a subspace.
   (a) Show that there exists another subspace $V \subset W$ such that $W = U \oplus V$.
   DEF We say V is a *complement* for U (in W).
   (b) Let V be a complement for U and let $\pi\colon W \to W/U$ be the quotient map. Show that the restriction of π to V is an isomorphism.
   (c) Conclude that if V_1, V_2 are both complements then $V_1 \simeq V_2$ (c.f. problem P2)
   REM A subspace will have many complements, while the quotient is "canonical".

Extra credit

6. (Structure of quotients) Let $V \subset W$ with quotient map $\pi \colon W \to W/V$.
   (a) Show that mapping $U \mapsto \pi(U)$ gives a bijection between (1) the set of subspaces of W containing V and (2) the set of subspaces of W/V.
   (b) (The universal property) Let Z be another vector space. Show that $f \mapsto f \circ \pi$ gives a linear bijection $\operatorname{Hom}(W/V, Z) \to \{g \in \operatorname{Hom}(W, Z) \mid V \subset \operatorname{Ker} g\}$.

7. For $f \colon \mathbb{R}^n \to \mathbb{R}$ the *Lipschitz constant* of f is the (possibly infinite) number

$$\|f\|_{\mathrm{Lip}} \stackrel{\mathrm{def}}{=} \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|} \mid x,y \in \mathbb{R}^n, x \neq y \right\}.$$

   Let $\operatorname{Lip}(\mathbb{R}^n) = \left\{ f \colon \mathbb{R}^n \to \mathbb{R} \mid \|f\|_{\mathrm{Lip}} < \infty \right\}$ be the space of *Lipschitz functions*.
   PRA Show that $f \in \operatorname{Lip}(\mathbb{R}^n)$ iff there is C such that $|f(x) - f(y)| \leq C\,|x-y|$ for all $x, y \in \mathbb{R}^n$.
   (a) Show that $\operatorname{Lip}(\mathbb{R}^n)$ is a vector space.
   (b) Let $\mathbb{1}$ be the constant function 1. Show that $\|f\|_{\mathrm{Lip}}$ descends to a function on $\operatorname{Lip}(\mathbb{R}^n)/\mathbb{R}\mathbb{1}$.
   (c) For $\bar{f} \in \operatorname{Lip}(\mathbb{R}^n)/\mathbb{R}\mathbb{1}$ show that $\left\|\bar{f}\right\|_{\mathrm{Lip}} = 0$ iff $\bar{f} = 0$.

Supplement: Infinite direct sums and products

CONSTRUCTION. *Let $\{V_i\}_{i \in I}$ be a (possibly infinite) family of vector spaces.*

(1) *The* direct product $\prod_{i \in I} V_i$ *is the vector space whose underlying space is* $\{f \colon I \to \bigcup_{i \in I} V_i \mid \forall i : f(i) \in V_i\}$ *with the operations of pointwise addition and scalar multiplication.*
(2) *The* direct sum $\bigoplus_{i \in i} V_i$ *is the subspace* $\left\{ f \in \prod_{i \in I} V_i \mid \#\left\{ i \mid f(i) \neq \underline{0}_{V_i} \right\} < \infty \right\}$ *of finitely supported functions.*

A. (Tedium)
   (a) Show that the direct product is a vector space
   (b) Show that the direct sum is a subspace.
   (c) Let $\pi_i \colon \prod_{j \in I} V_j \to V_i$ be the projection on the ith coordinate ($\pi_i(f) = f(i)$). Show that π_i are surjective linear maps.
   (d) Let $\sigma_i \colon V_i \to \prod_{i \in I} V_i$ be the map such that $\sigma_i(\underline{v})(j) = \begin{cases} \underline{v} & j = i \\ \underline{0} & j \neq i \end{cases}$.
   Show that σ_i are injective linear maps.

B. (Meat) Let Z be another vector space.
   (a) Show that $\bigoplus_{i \in I} V_i$ is the internal direct sum of the images $\sigma_i(V_i)$.
   (b) Suppose for each $i \in I$ we are given $f_i \in \operatorname{Hom}(V_i, Z)$. Show that there is a unique $f \in \operatorname{Hom}(\bigoplus_{i \in I} V_i)$ such that $f \circ \sigma_i = f_i$.
   (c) You are instead given $g_i \in \operatorname{Hom}(Z, V_i)$. Show that there is a unique $g \in \operatorname{Hom}(Z, \prod_i V_i)$ such that $\pi_i \circ g = g_i$ for all i.

C. (What a universal property can do) Let S be a vector space equipped with maps $\sigma_i' \colon V_i \to S$, and suppose the property of 5(b) holds (for every choice of $f_i \in \operatorname{Hom}(V_i, Z)$ there is a unique $f \in \operatorname{Hom}(S, Z)$ ...)
   (a) Show that each σ_i' is injective (hint: take $Z = V_j$, f_j the identity map, $f_i = 0$ if $i \neq j$).
   (b) Show that the images of the σ_i' span S.
   (c) Show that S is the internal direct sum of the S_i.

(d) (There is only one direct sum) Show that there is a unique isomorphism $\varphi\colon S \to \bigoplus_{i\in I} V_i$ such that $\varphi \circ \sigma_i' = \sigma_i$ (hint: construct φ by assumption, and a reverse map using the existence part of 5(b); to see that the composition is the identity use the uniqueness of the assumption and of 5(b), depending on the order of composition).

D. Now let P be a vector space equipped with maps $\pi_i'\colon P \to V_i$ such that 5(c) holds.
  (a) Show that π_i' are surjective.
  (b) Show that there is a unique isomorphism $\psi : P \to \prod_{i\in I} V_i$ such that $\pi_i \circ \psi = \pi_i'$.

Supplement: universal properties

E. A *free abelian group* is a pair (F,S) where F is an abelian group, $S \subset F$, and ("universal property") for any abelian group A and any (set) map $f\colon S \to A$ there is a unique group homomorphism $\bar{f}\colon G \to A$ such that $\bar{f}(s) = f(s)$ for any $s \in S$. The size $\#S$ is called the *rank* of the free abelian group.
  (a) Show that $(\mathbb{Z}, \{1\})$ is a free abelian group.
  (b) Show that $\left(\mathbb{Z}^d, \{\underline{e}_k\}_{k=1}^{d}\right)$ is a free abelian group.
  (c) Let $(F,S), (F',S')$ be free abelian groups and let $f\colon S \to S'$ be a bijection. Show that f extends to a unique isomorphism $\bar{f}\colon F \to F'$.
  (d) Let (F,S) be a free abelian group. Show that S generates F.
  (e) Show that every element of a free abelian group has infinite order.

Supplement: Lipschitz functions

DEFINITION. Let $(X,d_X), (Y,d_Y)$ be metric spaces, and let $f\colon X \to Y$ be a function. We say f is a *Lipschitz function* (or is "Lipschitz continuous") if for some C and for all $x,x' \in X$ we have

$$d_Y\left(f(x), f(x')\right) \le C d_X\left(x,x'\right) .$$

Write $\mathrm{Lip}(X,Y)$ for the space of Lipschitz continuous functions, and for $f \in \mathrm{Lip}(X,Y)$ write $\|f\|_{\mathrm{Lip}} = \sup\left\{\frac{d_Y(f(x),f(x'))}{d_X(x,x')} \mid x \neq x' \in X\right\}$ for its *Lipschitz constant*.

F. (Analysis)
  (a) Show that Lipschitz functions are continuous.
  (b) Let $f \in C^1(\mathbb{R}^n;\mathbb{R})$. Show that $\|f\|_{\mathrm{Lip}} = \sup\{|\nabla f(x)| : x \in \mathbb{R}^n\}$.
  (c) Generalize 7(a),(b),(c) to the case of $\mathrm{Lip}(X,\mathbb{R})$ where X is any metric space.
  (d) Show that $\mathrm{Lip}(X,\mathbb{R})/\mathbb{R}\mathbb{1}$ is a complete normed space for all metric spaces X.