Lior Silberman's Math 412: Problem Set 1 (due 12/9/2019)

Practice problems, any sub-parts marked "OPT" (optional) and supplementary problems are not for submission.

Practice problems

P1 Show that the map $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = x - 2y + z$ is a linear map. Show that the maps $(x,y,z) \mapsto 1$ and $(x,y,z) \mapsto x^2$ are non-linear.

P2 Let F be a field, X a set. Carefully show that pointwise addition and scalar multiplication endow the set F^X of functions from X to F with the structure of an F-vectorspace.

For submission

RMK This problem introduces a device for showing that sets of vectors are linearly independent. Make sure you understand how this argument works by solving 1(d),2(a),1(e).

1. Let V be a vector space, $S \subset V$ a set of vectors. A *minimal dependence* in S is an equality $\sum_{i=1}^{m} a_i \underline{v}_i = \underline{0}$ where $\underline{v}_i \in S$ are distinct, a_i are scalars not all of which are zero, and $m \geq 1$ is as small as possible so that such $\{a_i\}$, $\{\underline{v}_i\}$ exist.
   — It is implicit in the following that either S is independent or it has a minimal dependence. Make this explicit in your mind (don't write this bit up).
   (a) Find a minimal dependence among $\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix} \right\} \subset \mathbb{R}^3$.
   (b) Show that in a minimal dependence the a_i are all non-zero.
   (c) Suppose that $\sum_{i=1}^{m} a_i \underline{v}_i$ and $\sum_{i=1}^{m} b_i \underline{v}_i$ are minimal dependences in S, involving the exact same set of vectors. Show that there is a non-zero scalar c such that $a_i = cb_i$.
   (d) Let $T\colon V \to V$ be a linear map, and let $S \subset V$ be a set of (non-zero) eigenvectors of T, each corresponding to a distinct eigenvalue. Applying T to a minimal dependence in S obtain a contradiction to (c) and conclude that S is actually linearly independent.
   (*e) Let Γ be a group. The set $\mathrm{Hom}(\Gamma, \mathbb{C}^\times)$ of group homomorphisms from Γ to the multiplicative group of nonzero complex numbers is called the set of *quasicharacters* of Γ (the notion of "character of a group" has an additional, different but related meaning, which is not at issue in this problem). Show that $\mathrm{Hom}(\Gamma, \mathbb{C}^\times)$ is linearly independent in the space $\mathbb{C}^\Gamma$ of functions from Γ to $\mathbb{C}$.

2. Let $S = \{\cos(nx)\}_{n=0}^{\infty} \cup \{\sin(nx)\}_{n=1}^{\infty}$, thought of as a subset of the space $C(-\pi,\pi)$ of continuous functions on the interval $[-\pi,\pi]$.
   (a) Applying $\frac{d}{dx}$ to a putative minimal dependence in S obtain a different linear dependence of at most the same length, and use that to show that S is, in fact, linearly independent.
   (b) Show that the elements of S are an orthogonal system with respect to the inner product $\langle f,g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x$ (feel free to look up any trig identities you need). This gives a different proof of their independence.
   (c) Let $W = \mathrm{Span}_{\mathbb{C}}(S)$ (this is usually called "the space of trigonometric polynomials"; a typical element is $5 - \sin(3x) + \sqrt{2}\cos(15x) - \pi\cos(32x)$). Find a ordering of S so that the matrix of the linear map $\frac{d}{dx}\colon W \to W$ in that basis has a simple form.

3. (Matrices associated to linear maps) Let V, W be vector spaces of dimensions n, m respectively. Let $T \in \operatorname{Hom}(V, W)$ be a linear map from V to W. Show that there are ordered bases $B = \{\underline{v}_j\}_{j=1}^{n} \subset V$ and $C = \{\underline{w}_i\}_{i=1}^{m} \subset W$ and an integer $d \le \min\{n, m\}$ such that the matrix $A = (a_{ij})$ of T with respect to those bases satisfies $a_{ij} = \begin{cases} 1 & i = j \le d \\ 0 & \text{otherwise} \end{cases}$, that is has the form

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

(Hint1: study some examples, such as the matrices $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ and $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$) (Hint2: start your solution by choosing a basis for the image of T).

Extra credit: Finite fields

4. Let F be a field.
   (a) Define a map $\iota\colon (\mathbb{Z}, +) \to (F, +)$ by mapping $n \in \mathbb{Z}_{\ge 0}$ to the sum $1_F + \cdots + 1_F$ n times. Show that this is a group homomorphism.
   DEF If the map ι is injective we say that F is of *characteristic zero*.
   (b) Suppose there is a non-zero $n \in \mathbb{Z}$ in the kernel of ι. Show that the smallest positive such number is a prime number p.
   DEF In that case we say that F is of *characteristic p*.
   (c) Show that in that case ι induces an isomorphism between the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and a subfield of F. In particular, there is a unique field of p elements up to isomorphism.

5. Let F be a field with finitely many elements.
   (a) Carefully endow F with the structure of a vector space over $\mathbb{F}_p$ for an appropriately chosen p.
   (b) Show that there exists an integer $r \ge 1$ such that F has p^r elements.
   RMK For every prime power $q = p^r$ there is a field $\mathbb{F}_q$ with q elements, and two such fields are isomorphic. They are usually called *finite fields*, but also *Galois fields* after their discoverer.

Supplementary Problems I: A new field

A. Let $\mathbb{Q}(\sqrt{2})$ denote the set $\left\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\right\} \subset \mathbb{R}$.
   (a) Show that $\mathbb{Q}(\sqrt{2})$ is a $\mathbb{Q}$-subspace of $\mathbb{R}$.
   (b) Show that $\mathbb{Q}(\sqrt{2})$ is two-dimensional as a $\mathbb{Q}$-vector space. In fact, identify a basis.
   (*c) Show that $\mathbb{Q}(\sqrt{2})$ is a field.
   (**d) Let V be a vector space over $\mathbb{Q}(\sqrt{2})$ and suppose that $\dim_{\mathbb{Q}(\sqrt{2})} V = d$. Show that $\dim_{\mathbb{Q}} V = 2d$.

Supplementary Problems II: How physicists define vectors

Fix a field F.

B. (The general linear group)
   (a) Let $\mathrm{GL}_n(F)$ denote the set of invertible $n \times n$ matrices with coefficients in F. Show that $\mathrm{GL}_n(F)$ forms a group with the operation of matrix multiplication.
   (b) For a vector space V over F let $\mathrm{GL}(V)$ denote the set of invertible linear maps from V to itself. Show that $\mathrm{GL}(V)$ forms a group with the operation of composition.
   (c) Suppose that $\dim_F V = n$ Show that $\mathrm{GL}_n(F) \simeq \mathrm{GL}(V)$ (hint: show that each of the two group is isomorphic to $\mathrm{GL}(F^n)$.

C. (Group actions) Let G be a group, X a set. An *action* of G on X is a map $\cdot \colon G \times X \to X$ such that $g \cdot (h \cdot x) = (gh) \cdot x$ and $1_G \cdot x = x$ for all $g, h \in G$ and $x \in X$ (1_G is the identity element of G).
   (a) Show that matrix-vector multiplication $(g, \underline{v}) \mapsto g\underline{v}$ defines an action of $G = \mathrm{GL}_n(F)$ on $X = F^n$.
   (b) Let V be an n-dimensional vector space over F, and let $\mathcal{B}$ be the set of ordered bases of V. For $g \in \mathrm{GL}_n(F)$ and $B = \{\underline{v}_i\}_{i=1}^{\dim V} \in \mathcal{B}$ set $gB = \left\{\sum_{j=1}^n g_{ij}\underline{v}_i\right\}_{j=1}^n$. Check that $gB \in \mathcal{B}$ and that $(g, B) \mapsto gB$ is an action of $\mathrm{GL}_n(F)$ on $\mathcal{B}$.
   (c) Show that the action is *transitive*: for any $B, B' \in \mathcal{B}$ there is $g \in \mathrm{GL}_n(F)$ such that $gB = B'$.
   (d) Show that the action is *simply transitive*: that the g from part (b) is unique.

D. (From the physics department) Let V be an n-dimensional vector space, and let $\mathcal{B}$ be its set of bases. Given $\underline{u} \in V$ define a map $\phi_{\underline{u}} \colon \mathcal{B} \to F^n$ by setting $\phi_{\underline{u}}(B) = \underline{a}$ if $B = \{\underline{v}_i\}_{i=1}^n$ and $\underline{u} = \sum_{i=1}^n a_i \underline{v}_i$.
   (a) Show that $\alpha\phi_{\underline{u}} + \phi_{\underline{u}'} = \phi_{\alpha\underline{u}+\underline{u}'}$. Conclude that the set $\left\{\phi_{\underline{u}}\right\}_{\underline{u}\in V}$ forms a vector space over F.
   (b) Show that the map $\phi_{\underline{u}} \colon \mathcal{B} \to F^n$ is *equivariant* for the actions of B(a),B(b), in that for each $g \in \mathrm{GL}_n(F)$, $B \in \mathcal{B}$, $g\left(\phi_{\underline{u}}(B)\right) = \phi_{\underline{u}}(gB)$.
   (c) Physicists define a "covariant vector" to be an equivariant map $\phi \colon \mathcal{B} \to F^n$. Let Φ be the set of covariant vectors. Show that the map $\underline{u} \mapsto \phi_{\underline{u}}$ defines an isomorphism $V \to \Phi$. (Hint: define a map $\Phi \to V$ by fixing a basis $B = \{\underline{v}_i\}_{i=1}^n$ and mapping $\phi \mapsto \sum_{i=1}^n a_i \underline{v}_i$ if $\phi(B) = \underline{a}$).
   (d) Physicists define a "contravariant vector" to be a map $\phi \colon \mathcal{B} \to F^n$ such that $\phi(gB) = {}^t g^{-1} \cdot (\phi(B))$. Verify that $(g, \underline{a}) \mapsto {}^t g^{-1}\underline{a}$ defines an action of $\mathrm{GL}_n(F)$ on F^n, that the set Φ' of contravariant vectors is a vector space, and that it is naturally isomorphic to the dual vector space V' of V.

Supplementary Problems III: Fun in positive characteristic

E. Let F be a field of characteristic 2 (that is, $1_F + 1_F = 0_F$).
   (a) Show that for all $x, y \in F$ we have $x + x = 0_F$ and $(x+y)^2 = x^2 + y^2$.
   (b) Considering F as a vector space over $\mathbb{F}_2$ as in 5(a), show that the map $\mathrm{Frob} \colon F \to F$ given by $\mathrm{Frob}(x) = x^2$ is a linear map.
   (c) Suppose that the map $x \mapsto x^2$ is actually F-linear and not only $\mathbb{F}_2$-linear. Show that $F = \mathbb{F}_2$.
   RMK Compare your answer with practice problem 1.

F. (This problem requires a bit of number theory) Now let F have characteristic $p > 0$. Show that the *Frobenius endomorphism* $x \mapsto x^p$ is $\mathbb{F}_p$-linear.