## Lior Silberman's Math 412: Problem set 8, due 9/11/2016

### Practice: Norms

P1. Call two norms $\|\cdot\|_1, \|\cdot\|_2$ on $V$ *equivalent* if there are constants $c, C > 0$ such that for all $\underline{v} \in V$,

$$c\|\underline{v}\|_1 \le \|\underline{v}\|_2 \le C\|\underline{v}\|_1 .$$

(a) Show that this is an equivalence relation.

(b) Suppose the two norms are equivalent and that $\lim_{n\to\infty} \|\underline{v}_n\|_1 = 0$ (that is, that $\underline{v}_n \xrightarrow[n\to\infty]{\|\cdot\|_1} \underline{0}$). Show that $\lim_{n\to\infty} \|\underline{v}_n\|_2 = 0$ (that is, that $\underline{v}_n \xrightarrow[n\to\infty]{\|\cdot\|_2} \underline{0}$).

(*c) Show the converse of (b) also holds. In other words, two norms are equivalent iff they determine the same notion of convergence.

### Norms

1. Let $f(x) = x^2$ on $[-1, 1]$.
   (a) For $1 \le p < \infty$. Calculate $\|f\|_{L^p} = \left(\int_{-1}^{1} |f(x)|^p \, \mathrm{d}x\right)^{1/p}$.
   (b) Calculate $\|f\|_{L^\infty} = \sup\{|f(x)| : -1 \le x \le 1\}$. Check that $\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_\infty$.
   (c) Calculate $\|f\|_{H^2} = \left(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2 + \|f''\|_{L^2}^2\right)^{1/2}$.
   SUPP Show that the $H^2$ norm is equivalent to the norm $\left(\|f\|_{L^2}^2 + \|f''\|_{L^2}^2\right)^{1/2}$.

2. Let $A \in M_n(\mathbb{R})$. Write $\|A\|_p$ for its $\ell^p \to \ell^p$ operator norm.
   (a) Show $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$.
   (b) Show that $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$.
   RMK See below on *duality*.

3. The *spectral radius* of $A \in M_n(\mathbb{C})$ is the magnitude of its largest eigenvalue: $\rho(A) = \max\{|\lambda| \mid \lambda \in \mathrm{Spec}(A)\}$.
   (a) Show that for any norm $\|\cdot\|$ on $F^n$ and any $A \in M_n(F)$, $\rho(A) \le \|A\|$.
   (b) Suppose that $A$ is diagonable. Show that there is a norm on $F^n$ such that $\|A\| = \rho(A)$.
   (*c) Show that if $A$ is Hermitian then $\|A\|_2 = \rho(A)$.
   (d) Show that if $A, B$ are similar, and $\|\cdot\|$ is any norm in $\mathbb{C}^n$, then $\lim_{m\to\infty} \|A^m\|^{1/m} = \lim_{m\to\infty} \|B^m\|^{1/m}$ (in the sense that, if one limit exists, then so does the other, and they are equal).
   (**e) Show that for any norm on $\mathbb{C}^n$ and any $A \in M_n(\mathbb{C})$, we have $\lim_{m\to\infty} \|A^m\|^{1/m} = \rho(A)$.

4. The *Hilbert–Schmidt* norm on $M_n(\mathbb{C})$ is $\|A\|_{\mathrm{HS}} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}$.
   – Show that $\|A\|_{\mathrm{HS}} = \left(\mathrm{Tr}(A^\dagger A)\right)^{1/2}$.
   (a) Show that this is, indeed, a norm.
   (b) Show that $\|A\|_2 \le \|A\|_{\mathrm{HS}}$.

## Extra credit: Norms and constructions

5. (Direct sum) Let $\{(V_i, \|\cdot\|_i)\}_{i=1}^n$ be normed spaces, and let $1 \le p \le \infty$. For $\underline{v} = (\underline{v}_i) \in \bigoplus_{i=1}^n V_i$ define
$$\|\underline{v}\| = \left(\sum_{i=1}^n \|\underline{v}_i\|_i^p\right)^{1/p}.$$
Show that this defines a norm on $\bigoplus_{i=1}^n V_i$.
DEF This operation is called the $L^p$*-sum* of the normed spaces.
6. (Quotient) Let $(V, \|\cdot\|)$ be a normed space, and let $W \subset V$ be a subspace. For $\underline{v} + W \in V/W$ set $\|\underline{v}+W\|_{V/W} = \inf\{\|\underline{v}+\underline{w}\| : \underline{w} \in W\}$.
   (a) Show that $\|\cdot\|_{V/W}$ is 1-homogenous and satisfies the triangle inequality (a "seminorm").
   (b) Show that $\|v+W\|_{V/W} = 0$ iff $v$ is in the closure of $W$, so that $\|\cdot\|_{V/W}$ is a norm iff $W$ is closed in $V$.

For duality in norms see problems A, B. Norming tensor product spaces is complicated.

## Supplementary problems: Constructions

A. For $\underline{v} \in \mathbb{C}^n$ and $1 \le p \le \infty$ let $\|\underline{v}\|_p$ be as defined in class.
   (a) For $1 < p < \infty$ define $1 < q < \infty$ by $\frac{1}{p} + \frac{1}{q} = 1$ (also if $p = 1$ set $q = \infty$ and if $p = \infty$ set $q = 1$). Given $x \in \mathbb{C}$ let $y(x) = \frac{\bar{x}}{|x|}|x|^{p/q}$ (set $y = 0$ if $x = 0$), and given a vector $\underline{x} \in \mathbb{C}^n$ define a vector $\underline{y}$analogously.
      (i) Show that $\|\underline{y}\|_q = \|\underline{x}\|_p^{p/q}$.
      (ii) Show that for this particular choice of $vy$, $|\sum_{i=1}^n x_i y_i| = \|\underline{x}\|_p \|\underline{y}\|_q$
   (b) Now let $\underline{u}, \underline{v} \in \mathbb{C}^n$ and let $1 \le p \le \infty$. Show that $|\sum_{i=1}^n u_i v_i| \le \|\underline{u}\|_p \|\underline{v}\|_q$ (this is called *Hölder's inequality*).
   (c) Conlude that $\|\underline{u}\|_p = \max\left\{|\sum_{i=1}^n u_i v_i| \mid \|\underline{v}\|_q = 1\right\}$.
   (d) Show that $\|\underline{u}\|_p$ is a seminorm (hint: A(c)) and then that it is a norm.
   (e) Show that $\lim_{p\to\infty} \|\underline{v}\|_p = \|\underline{v}\|_\infty$ (this is why the supremum norm is usually called the $L^\infty$ norm).

B. Let $V$ be a normed space. The operator norm on $V^* = \mathrm{Hom}_{\mathrm{cts}}(V, F)$ is called the *dual norm*.
   (a) Let $V = \mathbb{R}^n$ and identify $V^*$ with $\mathbb{R}^n$ via the usual pairing. Show that the norm on $V^*$ dual to the $\ell^1$-norm is the $\ell^\infty$ norm and vice versa. Show that the $\ell^2$-norm is self-dual.
   (b) Use A(a),(b) to show that the dual to the $\ell^p$ norm on $\mathbb{R}^n$ is the $\ell^q$ norm where $\frac{1}{p} + \frac{1}{q} = 1$.
   (c) Let $U$ be another normed space and let $T : U \to V$ be bounded. Let $T' : V' \to U'$ be the algebraic dual map as defined in this course. Show that for every $\underline{v}^* \in V^* \subset V'$, $T'\underline{v}^* \in U^*$ (that is, it is continuous). We write $T^* : V^* \to U^*$ for the dual map restricted to continuous functionals.
   (d) Show that $T^*$ is itself bounded, in that $\|T^*\|_{V^* \to U^*} \le \|T\|_{U \to V}$.

C. A *seminorm* on a vector space $V$ is a map $V \to \mathbb{R}_{\geq 0}$ that satisfies all the conditions of a norm except that it can be zero for non-zero vectors.
   (a) Show that for any $f \in V'$, $\varphi(\underline{v}) = |f(\underline{v})|$ is a seminorm.
   (b) Construct a seminorm on $\mathbb{R}^2$ not of this form.
   (c) Let $\Phi$ be a family of seminorms on $V$ which is pointwise bounded. Show that $\bar{\varphi}(\underline{v}) = \sup\{\varphi(\underline{v}) \mid \varphi \in \Phi\}$ is again a seminorm.

**Supplementary problem: Continuity**

D. Let $V, W$ be normed vector spaces, equipped with the metric topology coming from the norm. Let $T \in \mathrm{Hom}_F(V, W)$. Show that the following are equivalent:
   (1) $T$ is continuous.
   (2) $T$ is continuous at zero.
   (3) $T$ is *bounded*: $\|T\|_{V \to W} < \infty$, that is: for some $C > 0$ and all $\underline{v} \in V$, $\|T\underline{v}\|_W \leq C \|\underline{v}\|_V$.
   Hint: the same idea is used in problem P1

## Supplementary problems: Completeness

E. Let $\{\underline{v}_n\}_{n=1}^{\infty}$ be a Cauchy sequence in a normed space. Show that $\{\|\underline{v}_n\|\}_{n=1}^{\infty} \subset \mathbb{R}_{\geq 0}$ is a Cauchy sequence.

F. Let $X$ be a set. For $1 \leq p < \infty$ set $\ell^p(X) = \{f \colon X \to \mathbb{C} \mid \sum_{x \in X} |f(x)|^p < \infty\}$, and also set $\ell^\infty(X) = \{f \colon X \to \mathbb{C} \mid f \text{ bounded}\}$.

(a) Show that for $f \in \ell^p(X)$ and $g \in \ell^q(X)$ we have $fg \in \ell^1(X)$ and $|\sum_{x \in X} f(x)g(x)| \leq \|f\|_p \|g\|_q$.

(b) Show that $\ell^p(X)$ are subspaces of $\mathbb{C}^X$, and that $\|f\|_p = (\sum_{x \in X} |f(x)|^p)^{1/p}$ is a norm on $\ell^p(X)$

(c) Let $\{f_n\}_{n=1}^{\infty} \subset \ell^p(X)$ be a Cauchy sequence. Show that for each $x \in X$, $\{f_n(x)\}_{n=1}^{\infty} \subset \mathbb{C}$ is a Cauchy sequence.

(d) Let $\{f_n\}_{n=1}^{\infty} \subset \ell^p(X)$ be a Cauchy sequence and let $f(x) = \lim_{n \to \infty} f_n(x)$. Show that $f \in \ell^p(X)$.

(e) Let $\{f_n\}_{n=1}^{\infty} \subset \ell^p(X)$ be a Cauchy sequence. Show that it is convergent in $\ell^p(X)$.

G. (The completion) Let $(X, d)$ be a metric space.

(a) Let $\{x_n\}, \{y_n\} \subset X$ be two Cauchy sequences. Show that $\{d(x_n, y_n)\}_{n=1}^{\infty} \subset \mathbb{R}$ is a Cauchy sequence.

DEF Let $(\tilde{X}, \tilde{d})$ denote the set of Cauchy sequences in $X$ with the distance $\tilde{d}(\underline{x}, \underline{y}) = \lim_{n \to \infty} d(x_n, y_n)$.

(b) Show that $\tilde{d}$ satisfies all the axioms of a metric except that it can be non-zero for distinct sequences.

(c) Show that the relation $\underline{x} \sim \underline{y} \iff \tilde{d}(\underline{x}, \underline{y}) = 0$ is an equivalence relation.

(d) Let $\hat{X} = \tilde{X}/\sim$ be the set of equivalence classes. Show that $\tilde{d} \colon \tilde{X} \times \tilde{X} \to \mathbb{R}_{\geq 0}$ descends to a well-defined function $\hat{d} \colon \hat{X} \times \hat{X} \to \mathbb{R}_{\geq 0}$ which is a metric.

(e) Show that $(\hat{X}, \hat{d})$ is a complete metric space.

DEF For $x \in X$ let $\iota(x) \in \hat{X}$ be the equivalence class of the constant sequence $x$.

(f) Show that $\iota \colon X \to \hat{X}$ is an isometric embedding with dense image.

(g) (Universal property) Show that for any complete metric space $(Y, d_Y)$ and any uniformly continuous $f \colon X \to Y$ there is a unique extension $\hat{f} \colon \hat{X} \to Y$ such that $\hat{f} \circ \iota = f$.

(h) Show that triples $(\hat{X}, \hat{d}, \iota)$ satisfying the property of (g) are unique up to a unique isomorphism.

Hint for F(d): Suppose that $\|f\|_p = \infty$. Then there is a finite set $S \subset X$ with $(\sum_{x \in S} |f(x)|^p)^{1/p} \geq \lim_{n \to \infty} \|f_n\| + 1$.