Practice

P1. Find the characteristic and minimal polynomial of each matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

P2. Show that $\begin{pmatrix} 0 & 1 & \alpha \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ are similar. Generalize to higher dimensions.

The Jordan Canonical Form

1. For each of the following matrices, (i) find the characteristic polynomial and eigenvalues (over the complex numbers), (ii) find the eigenspaces and generalized eigenspaces, (iii) find a Jordan basis and the Jordan form.

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

RMK I suggest computing by hand first even if you later check your answers with a CAS.

2. Suppose the characteristic polynomial of T is $x(x-1)^3(x-3)^4$.
    (a) What are the possible minimal polynomials?
    (b) What are the possible Jordan forms?

3. Let $T, S \in \text{End}_F(V)$.
    (a) Suppose that T, S are similar. Show that $m_T(x) = m_S(x)$.
    (b) Prove or disprove: if $m_T(x) = m_S(x)$ and $p_T(x) = p_S(x)$ then T, S are similar.

4. Let F be algebraically closed of characteristic zero. Show that every $g \in \text{GL}_n(F)$ has a square root, in that $g = h^2$ for some $h \in \text{GL}_n(F)$.

5. Let V be finite-dimensional, and let $\mathcal{A} \subset \text{End}_F(V)$ be an F-subalgebra, that is a subspace containing the identity map and closed under multiplication (composition). Suppose that $T \in \mathcal{A}$ is invertible in $\text{End}_F(V)$. Show that $T^{-1} \in \mathcal{A}$.

(extra credit problem on reverse)

Extra credit

6. (The additive Jordan decomposition) Let V be a finite-dimensional vector space, and let $T \in \text{End}_F(V)$.

   DEF An *additive Jordan decomposition* of T is an expression $T = S + N$ where $S \in \text{End}_F(V)$ is diagonable, $N \in \text{End}_F(V)$ is nilpotent, and S, N commute.

   (a) Suppose that F is algebraically closed. Separating the Jordan form into its diagonal and off-diagonal parts, show that T has an additive Jordan decomposition.

   (b) Let $S, S' \in \text{End}_F(V)$ be diagonable and suppose that S, S' commute. Show that $S + S'$ is diagonable.

   (c) Show that a nilpotent diagonable linear transformation vanishes.

   (d) Suppose that T has two additive Jordan decompositions $T = S + N = S' + N'$. Show that $S = S'$ and $N = N'$.

Supplementary problems

A. (conjugacy classes in $\text{GL}_n(F)$) Let F be a field, and let $G = \text{GL}_n(F)$.

   (a) Construct a bijection between conjugacy classes in G and certain Jordan forms. Note that the spectrum can lie in an extension field.

   (b) Enumerate the conjugacy classes in $\text{GL}_2(\mathbb{F}_p)$.

   (c) Enumerate the conjugacy classes of $\text{GL}_3(\mathbb{F}_p)$.