Math 412: Problem Set 4 (due 5/10/2016)

Practice

P1. Let U, V be vector spaces and let $U_1 \subset U$, $V_1 \subset V$ be subspaces.
(a) "Naturally" embed $U_1 \otimes V_1$ in $U \otimes V$.
(b) Is $(U \otimes V) / (U_1 \otimes V_1)$ isomorphic to $(U/U_1) \otimes (V/V_1)$?

P2. Let $(\cdot,\cdot)$ be a non-degenerate bilinear form on a finite-dimensional vector space U, defined by the isomorphism $g \colon U \to U'$ via $(\underline{u}, \underline{v}) \stackrel{\text{def}}{=} (g\underline{u})(\underline{v})$.
(a) For $T \in \operatorname{End}(U)$ define $T^\dagger = g^{-1} T' g$ where T' is the dual map. Show that $T^\dagger \in \operatorname{End}(U)$ satisfies $(\underline{u}, T\underline{v}) = \left(T^\dagger \underline{u}, \underline{v}\right)$ for all $\underline{u}, \underline{v} \in V$.
(b) Show that $(TS)^\dagger = S^\dagger T^\dagger$.
(c) Show that the matrix of $T^\dagger$ wrt an $(\cdot,\cdot)$-orthonormal basis is the transpose of the matrix of T in that basis.

Bilinear forms

In problems 1,2 we assume 2 is invertible in F, and fix F-vector spaces V, W.

1. (Alternating pairings and symplectic forms) Let V, W be vector spaces, and let $[\cdot,\cdot] : V \times V \to W$ be a bilinear map.
(a) Show that $(\forall \underline{u}, \underline{v} \in V : [\underline{u}, \underline{v}] = -[\underline{v}, \underline{u}]) \leftrightarrow (\forall \underline{u} \in V : [\underline{u}, \underline{u}] = 0)$ (Hint: consider $\underline{u} + \underline{v}$).
DEF A form satisfying either property is *alternating*. We now suppose $[\cdot,\cdot]$ is alternating.
PRAC Show that the *radical* $R = \{\underline{u} \in V \mid \forall \underline{v} \in V : [\underline{u}, \underline{v}] = 0\}$ of the form is a subspace.
(b) The form $[\cdot,\cdot]$ is called *non-degenerate* if its radical is $\{\underline{0}\}$. Show that setting $[\underline{u} + R, \underline{v} + R] \stackrel{\text{def}}{=} [\underline{u}, \underline{v}]$ defines a non-degenerate alternating bilinear map $(V/R) \times (V/R) \to W$.
RMK Note that you need to justify each claim, starting with "defines".

2. (Darboux's Theorem) Suppose now that V is finite-dimensional, and that $[\cdot,\cdot] : V \times V \to F$ is a non-degenerate alternating form.
DEF The *orthogonal complement* of a subspace $U \subset V$ is a set $U^\perp = \{\underline{v} \in V \mid \forall \underline{u} \in U : [\underline{u}, \underline{v}] = 0\}$.
PRAC Show that $U^\perp$ is a subspace of V.
(b) Show that the restriction of $[\cdot,\cdot]$ to U is non-degenerate iff $U \cap U^\perp = \{\underline{0}\}$.
(*c) Suppose that the conditions of (b) hold. Show that $V = U \oplus U^\perp$, and that the restriction of $[\cdot,\cdot]$ to $U^\perp$ is non-degenerate.
(d) Let $\underline{u} \in V$ be non-zero. Show that there is $\underline{u}' \in V$ such that $[\underline{u}, \underline{u}'] \neq 0$. Find a basis $\{\underline{u}_1, \underline{v}_1\}$to $U = \operatorname{Span}\{\underline{u}, \underline{u}'\}$ in which the matrix of $[\cdot,\cdot]$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
(e) Show that $\dim_F V = 2n$ for some n, and that V has a basis $\{\underline{u}_i, \underline{v}_i\}_{i=1}^n$ in which the matrix of $[\cdot,\cdot]$ is block-diagonal, with each 2×2 block of the form from (d).
RECAP Only even-dimensional spaces have non-degenerate alternating forms, and up to choice of basis, there is only one such form.

Tensor products

3. (Preliminary step) Let U, V be finite-dimensional.
(a) Construct a natural isomorphism $\operatorname{End}(U \otimes V) \to \operatorname{Hom}(U, U \otimes \operatorname{End}(V))$.
SUPP Generalize this to a natural isomorphism $\operatorname{Hom}(U \otimes V_1, U \otimes V_2) \to \operatorname{Hom}(U, U \otimes \operatorname{Hom}(V_1, V_2))$.

4. Let U, V be vector spaces with U finite-dimensional, and let $A \in \mathrm{Hom}(U, U \otimes V)$. Given a basis $\left\{\underline{u}_j\right\}_{j=1}^{\dim U}$ of U let $\underline{v}_{ij} \in V$ be defined by $A\underline{u}_j = \sum_i \underline{u}_i \otimes \underline{v}_{ij}$ and define $\mathrm{Tr} A = \sum_{i=1}^{\dim U} \underline{v}_{ii}$. Show that this definition is independent of the choice of basis.

Extra credit

5. (Partial traces) Let U, V real vector spaces be equipped with non-degenerate inner products.
   (a) Show that $\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle_{U \otimes V} \stackrel{\text{def}}{=} \langle u_1, u_2 \rangle_U \langle v_1, v_2 \rangle_V$ extends to an inner product on $U \otimes V$.
   (b) Let $A \in \mathrm{End}(U)$, $B \in \mathrm{End}(V)$. Show that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$ (for a definition of the adjoint see practice problem P2).
   (c) Let $P \in \mathrm{End}(U \otimes V)$, interpreted as an element of $\mathrm{Hom}(U, U \otimes \mathrm{End}(V))$ as in 3(a). Show that $(\mathrm{Tr}_U P)^\dagger = \mathrm{Tr}_U \left(P^\dagger\right)$.
   (*d) [Thanks to J. Karczmarek] Let $\underline{w} \in U \otimes V$ be non-zero, and let $P_{\underline{w}} \in \mathrm{End}(U \otimes V)$ be the orthogonal projection on $\underline{w}$. It follows from 3(a) that $\mathrm{Tr}_U P_{\underline{w}} \in \mathrm{End}(V)$ and $\mathrm{Tr}_V P_{\underline{w}} \in \mathrm{End}(U)$ are both Hermitian. Show that their non-zero eigenvalues are the same.

Supplementary problems

A. (Extension of scalars) Let $F \subset K$ be fields. Let V be an F-vectorspace.
   (a) Considering K as an F-vectorspace (see PS1), we have the tensor product $K \otimes_F V$ (the subscript means "tensor product as F-vectorspaces"). For each $x \in K$ set $x(\alpha \otimes \underline{v}) \stackrel{\text{def}}{=} (x\alpha) \otimes \underline{v}$. Show that this extends to an F-linear map $K \otimes_F V \to K \otimes_F V$ giving $K \otimes_F V$ the structure of a K-vector space. This construction is called "extension of scalars"
   (b) Let $B \subset V$ be a basis. Show that $\{1 \otimes \underline{v}\}_{\underline{v} \in B}$ is a basis for $K \otimes_F V$ as a K-vectorspace. Conclude that $\dim_K (K \otimes_F V) = \dim_F V$.
   (c) Let $V_N = \mathrm{Span}_{\mathbb{R}} \left(\{1\} \cup \{\cos(nx), \sin(nx)\}_{n=1}^N\right)$. Then $\frac{d}{dx}\colon V_N \to V_N$ is not diagonable. Find a different basis for $\mathbb{C} \otimes_{\mathbb{R}} V_N$ in which $\frac{d}{dx}$ is diagonal. Note that the elements of your basis are not "pure tensors", that is not of the form $af(x)$ where $a \in \mathbb{C}$and $f = \cos(nx)$ or $f = \sin(nx)$.

B. DEF: An *F-algebra* is a triple $(A, 1_A, \times)$ such that A is an F-vector space, $(A, 0_A, 1_A +, \times)$ is a ring, and (compatibility of structures) for any $a \in F$ and $x, y \in A$ we have $a \cdot (x \times y) = (a \cdot x) \times y = x \times (a \cdot y)$. Because of the compatibility from now on we won't distinguish the multiplication in A and scalar multiplication by elements of F.
   (a) Verify that $\mathbb{C}$ is an $\mathbb{R}$-algebra, and that $M_n(F)$ is an F-algebra for all F.
   (b) More generally, verify that if R is a ring, and $F \subset R$ is a subfield then R has the structure of an F-algebra. Similarly, that $\mathrm{End}_F(V)$ is an F-algebra for any vector space V.
   (c) Let A, B be F-algebras. Give $A \otimes_F B$ the structure of an F-algebra.
   (d) Show that the map $F \to A$ given by $a \mapsto a \cdot 1_A$ gives an embedding of F-algebars $F \hookrightarrow A$.
   (e) (Extension of scalars for algebras) Let K be an extension of F. Give $K \otimes_F A$ the structure of a K-algebra.
   (f) Show that for V finite-dimensional, $K \otimes_F \mathrm{End}_F(V) \simeq \mathrm{End}_K(K \otimes_F V)$.

C. The *center* $Z(A)$ of a ring is the set of elements that commute with the whole ring.
   (a) Show that the center of an F-algebra is an F-subspace, containing the subspace $F \cdot 1_A$.
   (b) Show that the image of $Z(A) \otimes Z(B)$ in $A \otimes B$ is exactly the center of that algebra.