Math 101 – SOLUTIONS TO WORKSHEET 29
THE RATIO TEST

(1) If the series converges, find its sum. Otherwise, state that it diverges.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+3}}{11^n}$
Solution: We rewrite the series as
$$\sum_{n=0}^{\infty} 3^3 \left(-\frac{3^2}{11}\right)^n = 27 \sum_{n=0}^{\infty} \left(-\frac{9}{11}\right)^n$$
we now see that we have a convergent geometric series, which sums to
$$= 27 \frac{1}{1-\left(-\frac{9}{11}\right)} = \boxed{\frac{27 \cdot 11}{20}}.$$

(b) $\sum_{n=1}^{\infty} (-1)^{n+2} \frac{3^{3n+2}}{11^n}$
Solution: We rewrite the series as
$$\sum_{n=1}^{\infty} 3^2 \left(-\frac{3^3}{11}\right)^n = 9 \sum_{n=1}^{\infty} \left(-\frac{27}{11}\right)^n .$$
This is a divergent geometric series (its ratio $-\frac{27}{11}$ has magnitude greater than 1).

(2) Decide whether the following series converge:

(a) $\sum_{n=0}^{\infty} \frac{n}{2^n}$
Solution: We have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \Big/ \frac{n}{2^n} = \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} = \frac{1}{2}\left(1+\frac{1}{n}\right) \xrightarrow[n\to\infty]{} \frac{1}{2} < 1$ so the series converges by the ratio test.

(b) $\sum_{n=0}^{\infty} \frac{n!}{2^n}$
Solution: We have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{2^{n+1}} \Big/ \frac{n!}{2^n} = \frac{(n+1)!}{n!} \cdot \frac{2^n}{2^{n+1}} = \frac{n+1}{2} \xrightarrow[n\to\infty]{} \infty > 1$ so the series diverges by the ratio test.

(c) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$
Solution: We have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$ so the series converges by the ratio test.

(d) For which values of x does $\sum_{n=0}^{\infty} nx^n$ converge?
Solution: Let $a_n = nx^n$. Then
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\,|x|^{n+1}}{n\,|x|^n} = \left(1+\frac{1}{n}\right)|x| \xrightarrow[n\to\infty]{} |x| .$$
By the ratio test, the series *converges* if $|x| < 1$ and *diverges* if $|x| > 1$. If $|x| = 1$ then $|a_n| = n\,|x|^n = n \xrightarrow[n\to\infty]{} \infty$ so the series *diverges* by the divergence test. We conclude that the series converges exactly when $|x| < 1$, that is for $x \in (-1, 1)$.

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