

**Math 101 – SOLUTIONS TO WORKSHEET 10**  
**INTEGRATION BY PARTS**

(1) Evaluate the following integrals:

(a)  $\int x^2 \cos x \, dx$

**Solution:** We integrate by parts twice, differentiating the term of the form  $x^k$

$$\begin{aligned} \int x^2 \cos x \, dx &= x^2 \sin x - \int (2x) \sin x \, dx \\ &= x^2 \sin x - \left[ (-2x \cos x) - \int (2)(-\cos x) \, dx \right] \\ &= x^2 \sin x + 2x \cos x - 2 \int \cos x \, dx \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C. \end{aligned}$$

(b)  $\int \log x \, dx$

**Solution:** Let  $u = \log x$ ,  $dv = 1 \, dx = dx$  so that  $v = x$ ,  $du = \frac{1}{x} \, dx$ . Integrating by parts, we get:

$$\begin{aligned} \int \log x \, dx &= x \log x - \int x \cdot \frac{1}{x} \, dx \\ &= x \log x - \int dx \\ &= x \log x - x + C. \end{aligned}$$

(c) (Final, 2013)  $\int_0^1 \arctan x \, dx =$

**Solution:** Let  $u = \arctan x$ ,  $dv = 1 \, dx = dx$  so that  $v = x$ ,  $du = \frac{dx}{1+x^2}$ . Integrating by parts, we get:

$$\begin{aligned} \int \arctan x \, dx &= x \arctan x - \int \frac{x \, dx}{1+x^2} \\ &\stackrel{w=1+x^2}{=} x \arctan x - \int \frac{\frac{1}{2} \, dw}{w} \quad (dw = 2x \, dx) \\ &= x \arctan x - \frac{1}{2} \log |w| + C \\ &= x \arctan x - \frac{1}{2} \log (1+x^2) + C. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^1 \arctan x \, dx &= \left[ x \arctan x - \frac{1}{2} \log (1+x^2) + C \right] = \arctan 1 - \frac{1}{2} \log 2 - \frac{1}{2} \log 1 \\ &= \frac{\pi}{4} - \frac{1}{2} \log 2. \end{aligned}$$

(2) Now let's play with our toolkit

(a) Evaluate  $\int \frac{\log x}{x} \, dx$

**Solution:** Let  $u = \log x$  so that  $du = \frac{1}{x} \, dx$ . We then have

$$\int \frac{\log x}{x} \, dx = \int u \, du = \frac{1}{2} u^2 + C = \frac{1}{2} \log^2 x + C.$$

(b) (Quiz 2015) Evaluate  $\int \frac{\log x}{x^3} dx$

**Solution:** This time we integrate by parts:

$$\begin{aligned}\int \frac{\log x}{x^2} dx &= \left(-\frac{1}{x}\right) \log x - \int \left(-\frac{1}{x}\right) \frac{1}{x} dx \\ &= -\frac{\log x}{x} + \int \frac{1}{x^2} dx \\ &= -\frac{\log x}{x} - \frac{1}{x} + C \\ &= -\frac{\log x + 1}{x} + C\end{aligned}$$

(c) (Final, 2010) Let  $g(x) = \int_0^1 (xe^t - t)^2 dt$ . Find the minimum value of  $g(x)$ .

**Solution:** We have

$$\begin{aligned}g(x) &= \int_0^1 (x^2 e^{2t} - 2xte^t + t^2) dt = \\ &= x^2 \int_0^1 e^{2t} dt + \int_0^1 t^2 - 2x \int_0^1 te^t dt \\ &= x^2 \left[ \frac{1}{2} e^{2t} \right]_{t=0}^{t=1} + \left[ \frac{t^3}{3} \right]_{t=0}^{t=1} - 2x [te^t]_{t=0}^{t=1} + 2x \int_0^1 e^t dt \\ &= \frac{e^2 - 1}{2} x^2 + \frac{1}{3} - 2ex + 2x [e^t]_{t=0}^{t=1} \\ &= \frac{e^2 - 1}{2} x^2 + 2(e - 1)x - 2ex + \frac{1}{3} \\ &= \frac{e^2 - 1}{2} x^2 - 2x + \frac{1}{3}.\end{aligned}$$

This is a concave-up parabola so has a unique minimum. We have  $g'(x) = (e^2 - 1)x - 2$  and the minimum is where  $g'(x) = 0$  that is where

$$x = \frac{2}{e^2 - 1}.$$

(d) Evaluate  $\int x^3 \log(x^2 + 1) dx$

**Solution:** We “peel off” an  $x$ , substituting  $u = x^2 + 1$  so that  $x^2 = u - 1$  and  $du = 2x dx$  so that

$$\begin{aligned}\int x^3 \log(x^2 + 1) dx &= \frac{1}{2} \int x^2 \log(x^2 + 1) \cdot 2x dx \\ &= \frac{1}{2} \int (u - 1) \log u du \\ &= \frac{1}{2} \left( \frac{1}{2} u^2 - u \right) \log u - \frac{1}{2} \int \left( \frac{1}{2} u^2 - u \right) \frac{1}{u} du \\ &= \frac{1}{4} (u^2 - 2u) \log u - \frac{1}{4} \int (u - 2) du \\ &= \frac{1}{4} (u^2 - 2u) \log u - \frac{1}{8} u^2 + \frac{1}{2} u + C \\ &= \frac{1}{4} ((x^2 + 1)^2 - 2(x^2 + 1)) \log(x^2 + 1) - \frac{1}{8} (x^2 + 1) (x^2 + 1 - 4) + C \\ &= \frac{1}{4} (x^4 - 1) \log(x^2 + 1) - \frac{1}{8} (x^4 - 2x^2) + C.\end{aligned}$$

**Solution:** We start by integrating by parts, using  $u = \log(x^2 + 1)$ ,  $dv = x^3 dx$  getting:

$$\begin{aligned}\int x^3 \log(x^2 + 1) dx &= \frac{1}{4}x^4 \log(x^2 + 1) - \frac{1}{4} \int x^4 \frac{2x}{x^2 + 1} dx \\ &= \frac{1}{4}x^4 \log(x^2 + 1) - \frac{1}{2} \int \frac{x^5}{x^2 + 1} dx \\ &= \frac{1}{4}x^4 \log(x^2 + 1) - \frac{1}{2} \int \left( \frac{x^5 + x^3}{x^2 + 1} - \frac{x^3 + x}{x^2 + 1} + \frac{x}{x^2 + 1} \right) dx \\ &= \frac{1}{4}x^4 \log(x^2 + 1) - \frac{1}{2} \int \left( x^3 - x + \frac{x}{x^2 + 1} \right) dx \\ &= \frac{1}{4}x^4 \log(x^2 + 1) - \frac{1}{8}x^4 + \frac{1}{4}x^2 - \frac{1}{4} \int \frac{du}{u} \quad u = x^2 + 1, du = 2x dx \\ &= \frac{1}{4}x^4 \log(x^2 + 1) - \frac{1}{8}x^4 + \frac{1}{4}x^2 - \frac{1}{4} \log u + C \\ &= \frac{1}{4}(x^4 - 1) \log(x^2 + 1) - \frac{1}{8}(x^4 - 2x^2) + C.\end{aligned}$$