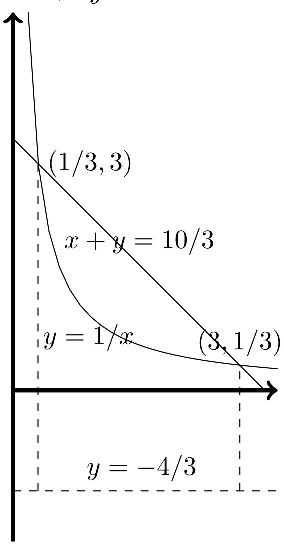
Math 101 – SOLUTIONS TO WORKSHEET 9
SOLIDS OF REVOLUTION, INTEGRATION BY PARTS

(1) Solids of revolution

(a) (Final 2014, variant) Find the volume of the solid generated by rotating the finite region bounded by $y = \frac{1}{x}$ and $3x + 3y = 10$ about the line $y = -\frac{4}{3}$. It will be useful to sketch the region first.



Solution: The intersection points are where $x + \frac{1}{x} = \frac{10}{3}$ that is where $x^2 - \frac{10}{3}x + 1 = 0$ that is where $x = \frac{10/3 \pm \sqrt{\frac{100}{9} - 4}}{2} = \frac{10 \pm \sqrt{64}}{6} = \frac{5 \pm 4}{3} = \frac{1}{3}, 3$. Setting $f(x) = \frac{10}{3} - x$ and $g(x) = \frac{1}{x}$ the region is $\left\{(x,y) \mid \frac{1}{3} \le x \le 3,\, g(x) \le y \le f(x)\right\}$; the cross-sections when revolving about the line $x = -\frac{4}{3}$ are annuli with inner radius $g(x) + \frac{4}{3}$, outer radius $f(x) + \frac{4}{3}$ and therefore area $\pi\left(\left(f(x) + \frac{4}{3}\right)^2 - \left(g(x) + \frac{4}{3}\right)^2\right)$ so the volume is:

$$\begin{aligned}
\pi \int_{x=1/3}^{x=1} \left(\left(\frac{10}{3} - x + \frac{4}{3}\right)^2 - \left(\frac{1}{x} + \frac{4}{3}\right)^2\right) \mathrm{d}x &= \pi \int_{x=1/3}^{x=3} \left(\frac{196}{9} - \frac{28}{3}x + x^2 - \frac{1}{x^2} - \frac{8}{3x} - \frac{16}{9}\right) \mathrm{d}x \\
&= \pi \int_{x=1/3}^{x=3} \left(20 - \frac{28}{3}x + x^2 - \frac{1}{x^2} - \frac{8}{3x}\right) \mathrm{d}x \\
&= \pi \left[20x - \frac{14}{3}x^2 + \frac{x^3}{3} + \frac{1}{x} - \frac{8}{3}\log|x|\right]_{x=1/3}^{x=3} \\
&= \pi \left[\left(60 - 42 + 9 + \frac{1}{3} - \frac{8}{3}\log 3\right) - \left(\frac{20}{3} - \frac{14}{27} + \frac{1}{81} + 3 - \frac{8}{3}\log\frac{1}{3}\right)\right] \\
&= \pi \left[18\frac{14}{81} - \frac{16}{3}\log 3\right] = 18\frac{14}{81}\pi - \frac{16\pi}{3}\log 3\,.
\end{aligned}$$

(b) The area between the y-axis, the curve $y = x^2$ and the line $y = 4$ is rotated about the y-axis. What is the volume of the resulting region?

Solution: Slicing perpendicular to the y-axis, we need to evaluate

$$\int_{y=0}^{y=4} \pi x^2\,\mathrm{d}y = \int_{y=0}^{y=4} \pi y\,\mathrm{d}y = \frac{\pi}{2}\left[y^2\right]_{y=0}^{y=4} = 8\pi\,.$$

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(2) Integrate by parts

(a) $\int xe^x \, \mathrm{d}x$

Solution: Let $u = x$, $\mathrm{d}v = e^x \, \mathrm{d}x$ so that $v = \int e^x \, \mathrm{d}x = e^x$. Then $\mathrm{d}u = \mathrm{d}x$ so that

$$\int xe^x \, \mathrm{d}x = \int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u = xe^x - \int e^x \, \mathrm{d}x = xe^x - e^x + C = (x-1)e^x + C\,.$$

(b) (Final, 2014) $\int x \log x \, \mathrm{d}x$

Solution: This time, let $u = \log x$, $\mathrm{d}v = x \, \mathrm{d}x$ so that $v = \frac{1}{2}x^2$ and $\mathrm{d}u = \frac{1}{x} \, \mathrm{d}x$. Integrating by parts, we get:

$$\begin{aligned} \int x \log x \, \mathrm{d}x &= \frac{1}{2}x^2 \log x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} \, \mathrm{d}x \\ &= \frac{1}{2}x^2 \log x - \frac{1}{2} \int x \, \mathrm{d}x \\ &= \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + C\,. \end{aligned}$$