29. Ratio Test (20/3/2017)

Goals:

(1) Use the ratio test.

Review: Suppose $q = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists.

- If $q > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $q < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Examples

("terms decay exponentially ⇓ use ratio test)

$\sum_{n=1}^{\infty}\frac{n}{2^n}$: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \Big/ \frac{n}{2^n} = \frac{2^n}{2^{n+1}}\cdot\frac{n+1}{n} = \frac{1}{2}\left(1+\frac{1}{n}\right) \xrightarrow[n\to\infty]{} \frac{1}{2} < 1$

so the series converges.

$\sum_{n=1}^{\infty}\frac{1}{n}$:

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n}{n+1} = 1$

$\sum_{n=1}^{\infty}\frac{1}{n^2}$:

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = 1$

limit can be 1 for both convergent & divergent series!

(common error: "geometric series with $q=1$ diverges" so $q=1 \Rightarrow$ divergence" but ratio test with $q=1$ is ambiguous.)

Math 101 – WORKSHEET 29
THE RATIO TEST

(1) If the series converges, find its sum. Otherwise, state that it diverges.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+3}}{11^n}$

This is a geometric series with ratio $\frac{(-1)\cdot 3^2}{11} = -\frac{9}{11}$, and $\frac{9}{11}<1$ so the series converges to

first term

$$\frac{3^3}{1-\left(-\frac{9}{11}\right)} = \frac{27}{1+\frac{9}{11}} = \frac{27\cdot 11}{20}.$$

1 ratio

(b) $\sum_{n=1}^{\infty} (-1)^{n+2} \frac{3^{3n+2}}{11^n}$

This is a geometric series with ratio $\frac{(-1)\, 3^3}{11} = -\frac{27}{11}$.

But $\frac{27}{11}>1$ so the series diverges.

Date: 20/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) Decide whether the following series converge:

(a) $\sum_{n=0}^{\infty} \frac{n}{2^n}$

reminder: $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n$

$(n+1)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n(n+1)$, $0! \overset{def}{=} 1$

(b) $\sum_{n=0}^{\infty} \frac{n!}{2^n}$

("$n!$ grows much faster than 2^n", so series must diverge - let's use ratio test)

$$\frac{(n+1)!}{2^{n+1}} \Big/ \frac{n!}{2^n} = \frac{(n+1)!}{n!} \cdot \frac{2^n}{2^{n+1}} = (n+1) \cdot \frac{1}{2} \xrightarrow[n\to\infty]{} \infty$$

so series diverges by ratio test

(c) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Here, $\frac{2^{n+1}}{(n+1)!} \Big/ \frac{2^n}{n!} = \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} = \frac{2}{n+1} \xrightarrow[n\to\infty]{} 0$

so the series ~~does~~ converges by ratio test.

(3) For which values of x do the following series converge:

(a) $\sum_{n=0}^{\infty} nx^n$ converge?

Here the terms of the series are $a_n = nx^n$.

So $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)x^{(n+1)}}{n\, x^n}\right| = \frac{n+1}{n}|x| = \left(1+\frac{1}{n}\right)|x| \xrightarrow[n\to\infty]{} |x|$.

So the series converges (absolutely) for $-1<x<1$, and diverges for $|x|>1$ by the ratio test.

At $x=\pm 1$, $|a_n| = n\cdot|(\pm1)^n| = n \xrightarrow[n\to\infty]{} \infty$ so $a_n \not\to 0$. By nth element test, the series diverges there.

final answer: $-1<x<1$.

(b) $\sum_{n=1}^{\infty} \frac{x^n}{n}$

$\left|\frac{x^{n+1}}{n+1} \Big/ \frac{x^n}{n}\right| = \left|\frac{x^{n+1}}{x^n}\cdot\frac{n}{n+1}\right| = |x|\cdot\frac{n}{n+1} \xrightarrow[n\to\infty]{} |x|$

so the series converges if $|x|<1$, diverges if $|x|>1$.

At $x=1$, the series is $\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges (p-series with $p=1$)

At $x=-1$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, whose terms alternate in sign, decrease in magnitude to 0, so converges by AST.

final answer: $-1\le x<1$

(c) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

$\left|\frac{x^{n+1}}{(n+1)^2} \Big/ \frac{x^n}{(n)^2}\right| = \left|\frac{x^{n+1}}{x^n}\cdot\left(\frac{n}{n+1}\right)^2\right| = |x|\cdot\left(\frac{1}{1+\frac{1}{n}}\right)^2 \xrightarrow[n\to\infty]{} |x|$.

Again, converge if $|x|<1$, diverge if $|x|>1$

If $|x|=1$, then $\left|\frac{x^n}{n^2}\right| = \frac{|x|^n}{n^2} = \frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series $p=2$),

so $\sum_{n=1}^{\infty}\frac{x^n}{n^2}$ converges absolutely at $x=\pm1$.

final answer: $-1\le x\le 1$