27. Alternating Series (15/3/2017)

Goals:

(1) Applying the test
(2) Tail estimates

Review of the (limit) comparison test

~~LCOOS~~ For positive series ($a_n \geq 0$), convergence is about rate of decay.

(1) Faster decay than convergent is convergent: $0 \leq a_n \leq b_n$, $\sum_n b_n < \infty$
$$\Downarrow$$
$$\sum a_n < \infty$$

(2) Slower decay than divergent is divergent. $0 \leq a_n \leq b_n$, $\sum_n a_n = \infty$
$$\Downarrow$$
$$\sum b_n = \infty$$

(3) Same rate of decay = same behaviour.

if $\lim_{n\to\infty} \frac{a_n}{b_n} \neq 0$, both $\sum a_n, \sum b_n$ diverge or both converge.

Today: "Alternating" series: series of the form $\sum_{n=1}^{\infty} (-1)^{n-1} A_n$, with $A_n \geq 0$

Examples $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

Fact: If A_n decrease, and $\lim_{n\to\infty} A_n = 0$ then $\sum_{n=1}^{\infty} (-1)^n A_n$ converges

[signs alternate]: $a_n = (-1)^n A_n$

Math 101 – WORKSHEET 27
ALTERNATING SERIES

1. Converge or Diverge?

(1) Determine, with explanation, whether the following series converge or diverge.

(a) (Alternating harmonic series) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

The terms alternate in sign, their magnitudes $A_n = \frac{1}{n}$ decrease with n, and $\lim_{n\to\infty} \frac{1}{n} = 0$ so by the AST the series converges

(b) $1-\frac{1}{4}+\frac{1}{3}-\frac{1}{16}+\frac{1}{5}-\frac{1}{36}+\frac{1}{7}-\frac{1}{64}+\frac{1}{9}-\frac{1}{100}+\frac{1}{11}-\frac{1}{144}+\cdots$

The terms alternate in sign, but don't decrease in magnitude:

$A_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ \frac{1}{n^2} & n \text{ even} \end{cases}$ so even terms smaller than neighbouring odd terms.

(c) (Final 2014) $\sum_{n=1}^{\infty} \frac{n\cos(\pi n)}{2^n}$

Here, terms alternate in sign $(\cos(\pi n) = (-1)^n)$. About magnitude.

let $f(x) = \frac{x}{2^x}$ $f'(x) = \frac{1}{2^x} - \frac{x\log 2}{2^x} = \frac{1-x\log 2}{2^x} < 0$ if $x > \frac{1}{\log 2}$, so $f(n)$ are eventually decreasing.

(d) (Final 2011) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$ (your answer will depend on p)

to show $\lim_{n\to\infty} \frac{n}{2^n} = 0$ use l'Hôpital

Date: 15/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Aside: 1(b): consecutive integers $(2n-1)$, $2n$ correspond to terms $\frac{1}{2n-1} - \frac{1}{(2n)^2}$

so then $2N$'th partial sum is

$$\sum_{n=1}^{N} \left(\frac{1}{2n-1} - \frac{1}{(2n)^2}\right),$$

if the series converges, so would

$$\left(1-\frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{16}\right) + \left(\frac{1}{5} - \frac{1}{36}\right) + \dots$$

$$\lim_{n\to\infty} \left(\frac{1}{(2n-1)} - \frac{1}{(2n)^2}\right) \Big/ \frac{1}{n} = \lim_{n\to\infty} \left(\frac{n}{2n-1} - \frac{1}{4n}\right) =$$

$$= \lim_{n\to\infty} \left(\frac{1}{2-\frac{1}{n}} - \frac{1}{4n}\right) = \frac{1}{2} \neq 0$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so by limit comparison test, $\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{(2n)^2}\right)$ diverges too.

- need to know whether AST applies
- need to know comparison tests

Could say: if $\sum (-1)^n a_n$ converges, since also $\sum_{n=1,\ \text{even}}^{\infty} \frac{1}{n^2}$ converge their sum $\sum_{n\ \text{odd}} \frac{1}{n}$ would converge. But it doesn't.

If : $\sum a_n$, $\sum_{a_n>0} a_n$, $\sum_{a_n<0} a_n$

If both converge - series converges

If one converges, one not - series diverges

If both diverse - indeterminate

Fact: In an alternatives series : $\sum_{n=1}^{\infty} (-1)^n A_n$, A_n decrease to 0,

Say $S = \sum_{n=1}^{\infty} (-1)^n A_n$, $s_N = \sum_{n=1}^{N} (-1)^n A_n$

Then $|S - s_N| < A_{N+1}$

error next term

2. Error estimates

(3)(a) It is known that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\log 2$. How many terms are needed for the error to be less than 0.01?

(check that AST applies - see 1(a))

By AST, error in using first N terms is the next term

If we use 99 terms, the error will be at most $\frac{1}{100}$.

(b) It is known that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots=\frac{\pi}{4}$. How many terms are needed for the error to be less than 0.001?

For $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)}$ terms alternate in sign, decrease to 0.

Have $\frac{1}{2n-1}\leq\frac{1}{1000}$ when $2n-1\geq 1000$, i.e. when $n\geq 500\frac{1}{2}$

so if we take first 500 terms, the error will be at most

$\frac{1}{2(\cancel{500})}$ $\frac{1}{2(501)-1}=\frac{1}{1001}<\frac{1}{1000}$