26. Comparison Test (13/3/2017)

Goals:

(1) Convergence is about rate of decay.
(2) Comparison by massaging.
(3) Limit comparison test.

Please ask questions!!

Integral test: $f(x)$ eventually positive, decreasing

$\Rightarrow$ test $\int_a^\infty f(x)dx$ instead of $\sum_{n=a}^\infty f(n)$

Main use: recognize $\int_a^\infty f(x)dx$ could be calculated

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Today: comparison of series

General philosophy: Have $\sum_{n=1}^\infty a_n$, $a_n \geq 0$. Suppose $\lim_{n\to\infty} a_n = 0$ (else diverge),

Question is: are a_n decaying fast enough.

Know: If $a_n = \frac{1}{n^p}$, $p>1$ ← fast enough

$a_n = \frac{1}{n^p}$ $p \leq 1$, ← not fast enough

If $a_n = a^{-n}$, $a>1$, fast enough.

Comparison test: For positive series, a series "better" than a convergent series is convergent, "worse" than a divergent series is divergent.

In symbols, suppose $0 \leq a_n \leq b_n$ (eventually)

Then: $\sum_{n=1}^\infty b_n$ converges $\Rightarrow$ $\sum_{n=1}^\infty a_n$ converges

$\sum_{n=1}^\infty a_n$ diverges $\Rightarrow$ $\sum_{n=1}^\infty b_n$ diverges

Math 101 – WORKSHEET 26
THE COMPARISON TEST

1. Comparison by Massaging

(1) Determine, with explanation, whether the following series converge or diverge.

(a) (Final 2014) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

$n^2+1 \le n^2+2n+1 = (n+1)^2$

Or: so $\frac{1}{\sqrt{n^2+1}} \ge \frac{1}{n+1}$

0) see decay like $\frac{1}{n}$ $\Rightarrow$ should diverge

1) Here: $n^2+1 \le n^2+n^2$ so $\frac{1}{\sqrt{n^2+1}} \ge \frac{1}{\sqrt{2n^2}} = \frac{1}{\sqrt{2}\cdot n} > 0$

$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{2}}n\right) = \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (harmonic series / p-series, $p=1$)

so by comparison, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+1}}$ diverges also.

(b) (Final 2013, variant) $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\frac{1}{121}+\cdots$

Here we are looking at

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Now, $2n-1 \ge 2n-n = n$ so $\frac{1}{(2n-1)^2} \le \frac{1}{n^2}$,

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series, $p=2>1$)

So by comparison $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ converges as well

Date: 13/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(c) (Final 2013) $\sum_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$

(terms roughly $\frac{n}{n^2} = \frac{1}{n}$, so expect divergence)

Have: $\frac{n+\sin n}{1+n^2} \geq \frac{n-1}{n^2+1} \overset{\text{if } n \geq 2}{\geq} \frac{n-n/2}{n^2+n^2} = \frac{\frac{1}{2}n}{2n^2} = \frac{1}{4n} > 0$

The series $\frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p=1$)

so by comparison $\sum_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ diverges too

(d) $1 + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^2} + \frac{1}{6^3} + \frac{1}{7^2} + \cdots$

~~Odd terms~~:

$a_n = \begin{cases} \frac{1}{n^2} & n \text{ odd} \\ \frac{1}{n^3} & n \text{ even} \end{cases} \leq \frac{1}{n^2}$

$a_n \geq 0$, ~~so~~ and $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, so by comparison, $\sum_{n=1}^{\infty} a_n$ converges too.

"converges"

$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (shorthand for "diverges")

Questions $\sqrt{n^2+1} \geq \sqrt{n^2} = n$ so $\frac{1}{\sqrt{n^2+1}} \leq \frac{1}{n}$

But $0 \leq \frac{1}{n}$

Question: Compare $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

(But not enough to say "look - they're all smaller!")

~~The~~ Can say: The n'th term of the series is $\frac{1}{m^2}$ where m is the n'th odd number. But then $m \geq n$ so $\frac{1}{m^2} \leq \frac{1}{n^2}$.

Facts Suppose $a_n, b_n \geq 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n} \neq 0$

then $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ converge or diverge together

(if $f(x), g(x) > 0$, $L = \lim_{n\to\infty} \frac{f(x)}{g(x)} \neq 0$, then $\int_a^\infty f\,dx$, $\int_a^\infty g\,dx$ both exist together)

2. Limit comparison test

(2) Determine, with explanation, whether the following series converge or diverge.

(a) (Final 2014) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

~~Here~~ $\frac{1}{\sqrt{n^2+1}} > 0$. Also, $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1 \neq 0$

~~By the Now~~ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

So by the limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges too

(b) (Final 2013, variant) $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \frac{1}{121} + \cdots$

Here $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ is a positive series.

$\lim_{n\to\infty} \frac{1}{(2n-1)^2} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} \left(\frac{n}{2n-1}\right)^2 = \lim_{n\to\infty} \left(\frac{1}{2-\frac{1}{n}}\right)^2 = \frac{1}{4} \neq 0$

~~So~~ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges (p-series, $p = 2 > 1$)

so by the limit comparison test $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ converges too.