17. Improper Integrals (10/2/2017)

Goals:

(1) Infinite regions of finite area:
   (a) Concept
   (b) Definition ("cutoff")
(2) Convergence:
   (a) A new kind of question
   (b) Comparison of integrals

Last time: Numerical Integration.

(1) Write approximations down
(2) Estimate error using a derivative bound (formulas given)
   (a) Given n, find upper bound for error.
   (b) Given upper bound for error, find n making this bound true.

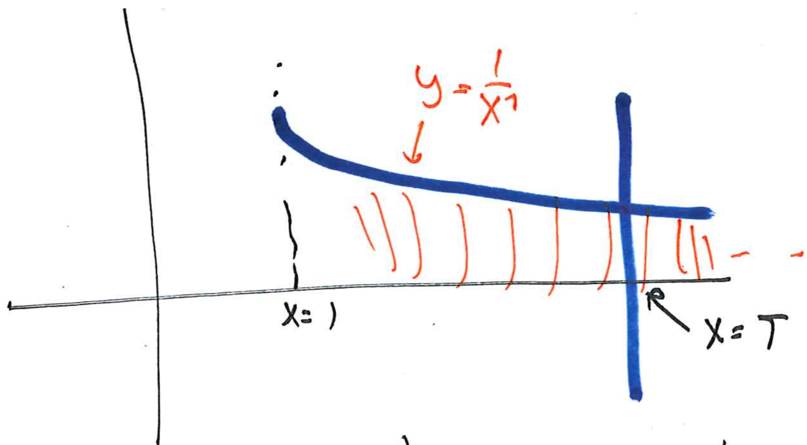
Example. On $[0,1]$ we have $f^{(2)}(x) = x(1-x)$. Find a bound for $\left|f^{(2)}(x)\right|$ on the interval.

$f^{(2)}(0)=0$, $f^{(2)}(1)=0$ need go beyond endpoints:

$0 \le x \le 1$, $\Rightarrow 0 \le x(1-x) \le 1$

$0 \le 1-x \le 1$

Question: What is the area under the curve $y = \frac{1}{x^2}$, to the right of the line $x = 1$?



Idea: ① cut off the region at $x = T$.

Area between $x=1$ and $x=T$ is $\int_{x=1}^{x=T} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^T = 1 - \frac{1}{T}$

② shift cutoff to infinity:

take limit at $T \to \infty$.

$$\lim_{T\to\infty} \left(1 - \frac{1}{T}\right) = 1.$$

Conclusion:
a) area is finite
b) area is equal to 1

Def: suppose f is ~~bounded~~ cts on $[a, \infty)$

Write $\int_a^\infty f(x)dx \overset{def}{=} \lim_{T\to\infty} \int_a^T f(x)dx$, if the limit exists

if it exists, say the integral converges if not, that it diverges.

Question: What if we had $y = \frac{1}{x}$ instead of $\frac{1}{x^2}$?

Now have $\int_1^T \frac{1}{x}dx = [\log x]_1^T = \log T \xrightarrow[T\to\infty]{} \infty$

so $\int_1^\infty \frac{1}{x}dx$ diverges.

Summary: "$\frac{1}{x^2}$ decays quickly", finite area under graph. "$\frac{1}{x}$ decays slowly": no area under its graph.

Math 101 – WORKSHEET 17
IMPROPER INTEGRALS

1. Improper at infinity

(1) For which values of p does $\int_1^\infty \frac{1}{x^p}\,dx$ converge? Diverge?

$$\int_1^T \frac{1}{x^p}dx \underset{p\neq 1}{=} \left[-\frac{1}{p-1}\frac{1}{x^{p-1}}\right]_1^T = \frac{1}{1-p} - \frac{1}{1-p}\frac{1}{T^{p-1}}$$

if $p>1$, $p-1>0$ $\frac{1}{T^{p-1}} \xrightarrow[T\to\infty]{} 0$, integral converges

if $p<1$, $p-1<0$ $\frac{1}{T^{p-1}} = T^{1-p} \xrightarrow[T\to\infty]{} \infty$, integral diverges.

Know: $\int_1^\infty \frac{dx}{x^p}$ converges iff $\boxed{p>1}$

(2) (Final, 2010) Evaluate $\int_{-\infty}^{-1} e^{2x}\,dx$. Simplify your answer as much as possible.

Solution: $\int_{-\infty}^{-1} e^{2x}dx = \lim_{T\to-\infty} \int_T^{-1} e^{2x}dx = \lim_{T\to-\infty}\left[\frac{1}{2}e^{2x}\right]_T^{-1} = \lim_{T\to\infty}\left(\frac{1}{2}e^{-2}-\frac{1}{2}e^{2T}\right)$

$= \frac{1}{2e^2} - 0 = \frac{1}{2e^2}.$

(also write as $\lim_{T\to\infty}\int_{-T}^{-1} e^{2x}dx$)

(3) Find a constant C such that $\int_{-\infty}^{+\infty} \frac{C\,\mathrm{d}x}{1+x^2} = 1$.

Need to compute $\int_0^\infty$, $\int_{-\infty}^0$ separately.

(break in middle - only one impropriety at a time

(4) We study $\int_{-\infty}^{+\infty} x\,\mathrm{d}x$.

(a) Evaluate $\int_{-T}^{T} x\,\mathrm{d}x$.

(b) Evaluate $\lim_{T\to\infty} \int_{-T}^{T} x\,\mathrm{d}x$.

(c) Does the integral converge?

no.

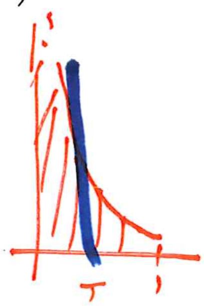
Consider $\int_{-T}^{T^2} x\,dx \xrightarrow[T\to\infty]{} \infty$

$\int_{-T^2}^{T} x\,dx \xrightarrow[T\to\infty]{} -\infty$

(5) (Final, 2009) For what values of p does $\int_e^\infty \frac{\mathrm{d}x}{x(\log x)^p}$ converge?

2. Improper at finite points

(6) For which values of p does $\int_0^1 \frac{\mathrm{d}x}{x^p}$ converge?



cut off at T: $\int_T^1 \frac{dx}{x^p} = \left[\frac{1}{p-1} - \frac{1}{p-1}\frac{1}{T^{p-1}}\right]$ $p \neq 1$

Now if $p<1$ $\frac{1}{T^{p-1}} = T^{1-p} \xrightarrow[T\to 0]{} 0$

if $p>1$ $\frac{1}{T^{p-1}} \xrightarrow[T\to 0]{} \infty$

Converge if $p<1$
diverge if $p \geq 1$

(7) (Math 103 Final, 2013) Evaluate the integral if it exists, otherwise show that it doesn't: $I = \int_0^2 \frac{\mathrm{d}x}{1-x^2}$.