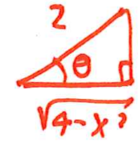
Math 101 – WORKSHEET 12
TRIGONOMETRIC SUBSTITUTION

1. Trig substitution

(1) (Final, 2014) Evaluate $\int \sqrt{4-x^2}\,\mathrm{d}x$



Substitute $x=2\sin\theta$, $dx=2\cos\theta\,d\theta$

$\sin\theta = \frac{x}{2}$

$x=2\sin\theta$

get $\int\sqrt{4-x^2}\,dx = \int\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta =$

$=2\int\sqrt{4}\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = 4\int\cos^2\theta\,d\theta = \dots = 4\left(\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right)+C$

$=2\theta+2\sin\theta\cos\theta+C = 2\,\cancel{\sin}\arcsin\left(\frac{x}{2}\right)+x\underbrace{\frac{\sqrt{4-x^2}}{2}}_{=\cos\theta}+C = 2\arcsin\left(\frac{x}{2}\right)+\frac{1}{2}x\sqrt{4-x^2}+C$

Alternative: want $\sqrt{4-(\,)^2}$ to be a square, use $x=2\sin\theta$ so that $\sqrt{4-x^2}=\sqrt{4-4\sin^2\theta}=$.

(2) (Final, 2013) Evaluate $\int_{-1}^{1}\frac{\mathrm{d}x}{(x^2+1)^3}$

Hint: $\tan^2\theta+1=\sec^2\theta$

We use $x=\tan\theta$, $dx=\sec^2\theta\,d\theta$, $\tan\left(\frac{\pi}{4}\right)=1$, $\tan\left(-\frac{\pi}{4}\right)=-1$:

$\int_{x=-1}^{x=1}\frac{dx}{(x^2+1)^3} = \int_{\theta=-\pi/4}^{\theta=\pi/4}\frac{1}{(1+\tan^2\theta)^3}\cdot\sec^2\theta\,d\theta = \int_{\theta=-\pi/4}^{\theta=+\pi/4}\frac{\sec^2\theta}{\sec^6\theta}\,d\theta = \int_{-\pi/4}^{\pi/4}\cos^4\theta\,d\theta$

$\uparrow$ $1+\tan^2\theta=\sec^2\theta$

$=\int_{-\pi/4}^{\pi/4}\left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta = \frac{1}{4}\int_{\alpha=-\pi/2}^{\alpha=\pi/2}\left(1+2\cos\alpha+\cos^2\alpha\right)\frac{1}{2}\,d\alpha = \left[\frac{1}{8}\alpha+\frac{1}{4}\sin\alpha\right]_{-\pi/2}^{\pi/2}+\frac{1}{8}\int_{-\pi/2}^{+\pi/2}\cos^2\alpha\,d\alpha$

$\uparrow$ $2\theta=\alpha$

$=\frac{\pi}{8}+\frac{1}{2}+\frac{1}{8}\int_{-\pi/2}^{+\pi/2}\left(\frac{1+\cos(2\alpha)}{2}\right)d\alpha = \frac{\pi}{8}+\frac{1}{2}+\frac{1}{16}\left[\alpha+\frac{\sin(2\alpha)}{2}\right]_{\alpha=-\pi/2}^{+\pi/2} =$

Date: 30/1/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$=\frac{\pi}{8}+\frac{1}{2}+\frac{\pi}{16} = \frac{3\pi}{16}+\frac{1}{2}.$

12. Trigonometric Substitution (30/1/2017)

Quiz 1 version A2 Problem 2(a) - Marking scheme error

→ regrade request

Goals.

(1) New integration technique.
   (a) Recalling trig identities
   (b) Trig substitutions
   (c) The right triangle trick
   (d) Completing the square
(2) Idea: where do we gain?

Last time: $\int \sin^n\theta \cos^m\theta \, d\theta$.

Today: convert integrals to this form.

Example: Compute $\int \sqrt{1-x^2}\, dx$

Solution: Recall $1-(\sin\theta)^2 = (\cos\theta)^2$

We want to set $x = \sin\theta$, $dx = \cos\theta \, d\theta$ ← find trig substitution

With this substitution, $\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2\theta} \cdot \cos\theta \, d\theta = \int \cos\theta \cdot \cos\theta \, d\theta$

algebraic simplification

$$= \int \cos^2\theta \, d\theta = \int \frac{1+\cos(2\theta)}{2} d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$

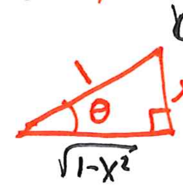
trig integral; discussed last time

To go back to x, use $\theta = \arcsin x$, $\sin(2\theta) = 2\sin\theta\cos\theta$
$= 2x\sqrt{1-x^2}$

So

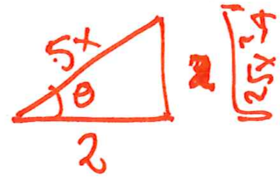
$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$$

choose sides to make $\sin\theta = x$



(3) (105 Final, 2012) Evaluate the indefinite integral
$$\int \frac{\sqrt{25x^2-4}}{x}\,dx$$

$\sec^2\theta - 1 = \tan^2\theta$



$\leftarrow \sec\theta = \frac{5x}{2}$

Want: $x = \frac{2}{5}\sec\theta$ or $\sqrt{25x^2-4} = \sqrt{4}\cdot\sqrt{\frac{25}{4}x^2-1}$

$\frac{25x^2}{4}$

want $\frac{25}{4}x^2 = \sec^2\theta$ so

$x = \frac{2}{5}\sec\theta$

$x = \frac{2}{5}\sec\theta, \quad dx = \frac{2}{5}\sec\theta\tan\theta$

$$\int \frac{\sqrt{25x^2-4}}{x}\,dx = \int \frac{\sqrt{4\sec^2\theta-4}}{\frac{2}{5}\cancel{\sec\theta}}\,\frac{2}{5}\cancel{\sec\theta}\tan\theta\,d\theta = \frac{2}{\cancel{5}}\int \tan^2\theta\,d\theta$$

$$= 2\int(\sec^2\theta - 1)\,d\theta = 2\tan\theta - 2\theta + C = 2\frac{\sqrt{25x^2-4}}{2} - 2\arccos\left(\frac{2}{5x}\right) + C$$

Summary:

To simplify a^2-x^2 use $x=a\sin\theta$ $(1-\sin^2\theta=\cos^2\theta)$

a^2+x^2 use $x=a\tan\theta$ $(1+\tan^2\theta=\sec^2\theta)$

x^2-a^2 use $x=a\sec\theta$ $(\sec^2\theta-1=\tan^2\theta)$

use  to return from θ to x.