5. Indefinite integrals (13/1/2017)

Goals.

(1) Net change theorem.
(2) Anti-derivatives and indefinite integrals.
(3) Computing definite integrals.

Last Time: FTC.

(1) Differentiating integrals.
(2) Definite integrals via anti-derivatives.

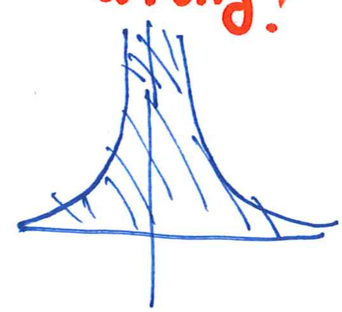
Part 2: If f is cts on $[a,b]$, F is any anti-derivative of f then $\int_a^b f(x)\,dx = F(b) - F(a)$

Example: $\int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{x=1}^{x=2} = \left[-\frac{1}{2} - (-1)\right] = \frac{1}{2}$

$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$ so $\left(-\frac{1}{x}\right)' = \frac{1}{x^2}$

Example: $\int_{-1}^{1} \frac{1}{x^2}\,dx \neq \left[-\frac{1}{x}\right]_{x=-1}^{x=1} = \left[-\frac{1}{1} - \left(-\frac{1}{-1}\right)\right] = -2$ ← clearly wrong!

$f(x) = \frac{1}{x^2}$ discontinuous on $[-1,1]$ so thm does not apply.



Lesson: check f is cts on $[a,b]$ first!

Theorem (Net change). *Suppose f' is continuous. Then* $\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a)$.

(1) (Net change theorem)

(a) A particle moves with velocity $v(t) = \pi \sin(\pi t)$. What is its displacement between the times $t = 0$ and $t = 2$?

By net change thm,

$$x(2) - x(0) = \int_0^2 v(t)\,dt = \int_0^2 \pi \sin(\pi t)\,dt$$

need anti-derivative for $\pi \sin(\pi t)$: $(\cos t)' = -\sin t$, so $(\cos(\pi t))' = -\pi \sin(\pi t)$

so $(-\cos(\pi t))' = \pi \sin(\pi t)$

so $\int_0^2 \pi \sin(\pi t)\,dt = \left[-\cos(\pi t)\right]_0^2 = -\cos(2\pi) + \cos(0) = 0$

(b) What is the total distance covered by the particle?

$\sin(\pi t) \begin{cases} \geq 0 & \text{if } 0 \leq t \leq 1 \\ \leq 0 & 1 \leq t \leq 2 \end{cases}$ so the distance covered is

$$\int_0^1 \pi \sin(\pi t)\,dt + (-1)\int_1^2 \pi \sin(\pi t)\,dt = \left[-\cos(\pi t)\right]_0^1 - \left[-\cos(\pi t)\right]_1^2 =$$

$$= \underbrace{-\cos \pi + \cos 0}_{2} + \underbrace{(\cos(2\pi) - \cos(\pi))}_{2} = 4$$

Date: 13/1/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Anti-derivatives

Notation: If f is cts, write $\int f\,dx$ or $\int f(x)\,dx$ for the general anti-derivative of f.

Call this the indefinite integral of f.

Example:

$$\int 1\,dx = x + C$$

$$\int 0\,dx = C$$

$$\int (f+g)\,dx = \int f\,dx + \int g\,dx$$

$$\int c\cdot f(x)\,dx = c\int f(x)\,dx$$

Short Table of Indefinite Integrals

(1) $(p \neq -1) \int x^p \,\mathrm{d}x = \frac{x^{p+1}}{p+1} + C$

(2) $\int \frac{\mathrm{d}x}{x} = \log|x| + C$

(3) $\int e^x \,\mathrm{d}x = e^x + C$

(4) (basic trig)

- (a) $\int \sin x \,\mathrm{d}x = -\cos x + C$
- (b) $\int \cos x \,\mathrm{d}x = \sin x + C$
- (c) $\int \frac{\mathrm{d}x}{\cos^2 x} = \tan x + C$

(5) (inverse trig)

- (a) $\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C$
- (b) $\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$

remark: if $x>0$, $(\log x)' = \frac{1}{x}$
if $x<0$ $(\log(-x))' = -\frac{1}{(-x)} = \frac{1}{x}$ $\Bigg\}$ $(\log|x|)' = \frac{1}{x}$ works for $x>0$ and $x<0$

(2) Find the indefinite integrals

(a) For $n \neq -1, \int x^n \, \mathrm{d}x =$ $\frac{1}{n+1} x^{n+1} + C$

$(x^{n+1})' = (n+1) x^n$ so $\left(\frac{1}{n+1} x^{n+1}\right)' = x^n$

(b) $\int \left(\frac{1}{2}x^{3/2} - e^{-x/3} + 7\right) \mathrm{d}x =$ $\frac{1}{5} x^{5/2} + 3e^{-x/3} + 7x + C$

$(x^{5/2})' = \frac{5}{2} x^{3/2}$ $(e^x)' = e^x$

$(e^{-x/3})' = -\frac{1}{3} e^{-x/3}$

so $\left(\frac{1}{5} x^{5/2}\right)' = \frac{1}{2} x^{3/2}$ $(3e^{-x/3})' = -e^{-x/3}$

(c) $\int_4^9 \left(x^{5/2} + e^{2x}\right) \mathrm{d}x =$ $\left[\frac{2}{7} x^{7/2} + \frac{1}{2} e^{2x}\right]_{x=4}^{x=9} = \ldots$

$\int (x^{5/2} + e^{2x}) \, dx = \frac{2}{7} x^{7/2} + \frac{1}{2} e^{2x} + C$

$= \frac{2}{7} 9^{7/2} + \frac{1}{2} e^{18} - \frac{2}{7} 4^{7/2} - \frac{1}{2} e^{8} = \ldots$

(d) $\int x \left(e^{x^2} + 1\right) \mathrm{d}x =$