Lior Silberman's Math 539: Problem Set 2 (due 24/2/2016)

Dirichlet Characters

0. List all Dirichlet characters mod 15 and mod 16. Determine which are primitive.

1. Let χ be a non-principal Dirichlet character mod q, and let $n_\chi = \min\{n \geq 1 \mid \chi(n) \neq 1\}$. Show that n_χ is prime.

2. (Uniqueness of the conductor) Let $f\colon \mathbb{Z} \to \mathbb{C}$ satisfy $f(a+N) = f(a)$ and $f(a) = 0$ whenever $(a,N) > 1$. Call q a *period* of f if $f(a) = f(b)$ whenver $a \equiv b\,(q)$ and both a,b are prime to N.
   (a) Suppose q_1, q_2 are periods of f, and let $q = \gcd(q_1,q_2)$. Show that q is a period as well (hint: given a,b prime to N such that $a \equiv b\,(q)$ show that there are $x,y \in \mathbb{Z}$ such that $b - a = xq_1 + yq_2$ with $a + xq_1$ prime to N).
   (b) Show that there is a unique $q = q(f)$ and a unique $g\colon \mathbb{Z} \to \mathbb{C}$ which is q-periodic, supported on integers prime to q and primitive (the only period of g is q) such that $f(n) = g(n)$ for all n prime to N.

3. Fix $q > 1$.
   (a) Let χ be a non-principal Dirichlet character mod q. Show that $\sum_p \frac{\chi(p)}{p}$ converges.
   (b) Let $(a,q) = 1$. Show that $\sum_{p\equiv a(q), p\leq x} \frac{1}{p} = \frac{1}{\varphi(q)} \log\log x + O(1)$
   (*c) Improve the error term to $C + O\left(\frac{1}{\log x}\right)$.

Counting with characters

Fix an odd prime p.

4. (The quadratic character) Recall that the group $U = (\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic of order $p-1$.
   (a) Show that the map $x \to x^2$ is a group homomorphism $U \to U$ with kernel of size 2, hence that the set S of squares mod p has order $\frac{p-1}{2}$.
   (b) Note that $U/S \simeq C_2$ and obtain the *quadratic character* (Legendre symbol), a group homomorphism $\left(\frac{\cdot}{p}\right) : U \to \{\pm 1\}$ such that $\left(\frac{a}{p}\right) = 1$ iff $x^2 = a$ is solvable in U.
   (c) Write $\chi(a)$ for this character, and extend it to $\mathbb{Z}/p\mathbb{Z}$ by setting $\chi(0) = 0$. Show that $1 + \chi(a)$ is the number of solutions to $x^2 = a$ in $\mathbb{Z}/p\mathbb{Z}$.
   (d) Consider the equation $x^2 + y^2 = c$ in $\mathbb{Z}/p\mathbb{Z}$ ($c \neq 0$). Show that its number of solutions is $\sum_{a+b=c} (1+\chi(a))(1+\chi(b))$. Use the identity $\chi(a)\chi(b) = \chi\left(\frac{a}{c-a}\right)$ (if $a \neq c$) to show that $\sum_{a+b=c} \chi(a)\chi(b) = -\chi(-1)$ and hence that the equation has $p - \chi(-1)$ solutions.

5. (A linearly uniform but quadratically non-uniform set) Fix a smooth cutoff function $\varphi\colon \mathbb{R}/\mathbb{Z} \to [0,1]$ supported on $[-\varepsilon-\delta, \varepsilon+\delta]$ and identically equal to 1 on $[-\varepsilon, \varepsilon]$. For each prime p define $F(x) = \varphi\left(\frac{x^2}{p}\right)$ (this roughly locates those x such that x^2 has a representative within εp of zero).
   (a) Show that $\left|\frac{1}{p}\sum_{x(p)} F(x) - \varepsilon\right| \leq \delta + O\left(\frac{1}{\sqrt{p}}\right)$ and that for $k \not\equiv 0\,(p)$, $\frac{1}{p}\sum_{x(p)} F(x) e_p(-kx) = O_\varphi(\frac{1}{\sqrt{p}})$.
   (b) Let $A_\varepsilon \subset \mathbb{Z}/p\mathbb{Z}$ be the set of x such that x^2 has a representative within εp of zero. Show that A_ε has density $\varepsilon + O(\delta)$ and has $\varepsilon^3 p^2 + O(\delta)p^2 + O_{\delta,\varepsilon}(p^{3/2})$ 3-APs.
   (c) Establish the identity $x^2 - 3(x+d)^2 + 3(x+2d)^2 - (x+3d)^2 = 0$ and conclude that if $x, x+d, x+2d \in A_{\varepsilon/7}$ then $x+3d \in A_\varepsilon$ and hence that the number of 3APs in A_ε is $\geq C\varepsilon^3 p^2$.
   RMK If the count of 4APs was controlled by Fourier coefficients, we'd exepct $\varepsilon^4 p^2$ 4APs, and as $\varepsilon \to 0$ this is a very different number.

Fourier analysis on the circle

6. (Basics of Fourier series)
   (a) Let $D_N(x) = \sum_{|k|\leq N} e(kx)$ be the Dirichlet kernel. Show that $\int_0^1 |D_N(x)|\,\mathrm{d}x \gg \log N$.
   (b) Let $F_N(x) = \sum_{|k|<N}\left(1 - \frac{|k|}{N}\right) e(kx)$ be the Fejér kernel. Show that for $\delta \leq |x| \leq \frac{1}{2}$, we have $|F_N(x)| \leq \frac{1}{N\sin^2(\pi\delta)}$ so that for $f \in L^1(\mathbb{R}/\mathbb{Z})$,
   $$\lim_{N\to\infty} \int_{\delta\leq|x|\leq\frac{1}{2}} |f(x)|\,|F_N(x)|\,\mathrm{d}x = 0\,.$$
   (c) In class we showed that "smoothness implies decay": if $f \in C^r(\mathbb{R}/\mathbb{Z})$ then for $k \neq 0$, $\left|\hat{f}(k)\right| \ll_r \|f\|_{C^r} |k|^{-r}$. Show the following partial converse: if $\left|\hat{f}(k)\right| = O(k^{-r-\varepsilon})$ then $\sum_{k\in\mathbb{Z}} \hat{f}(k)e(kx) \in C^{r-1}(\mathbb{R}/\mathbb{Z})$.

7. (The Basel problem) Let $f(x)$ be the $\mathbb{Z}$-periodic function on $\mathbb{R}$ such that $f(x) = x^2$for $|x| \leq \frac{1}{2}$.
   (a) Find $\hat{f}(k)$ for $k \in \mathbb{Z}$.
   (b) Show that $\zeta(2) = \frac{\pi^2}{6}$.
   (c) Apply *Parseval's identity* $\|f\|_{L^2(\mathbb{R}/\mathbb{Z})} = \left\|\hat{f}\right\|_{L^2(\mathbb{Z})}$ to evaluate $\zeta(4)$.

8. Let $\varphi \in \mathcal{S}(\mathbb{R})$.
   (a) Let $c \in L^2(\mathbb{Z}/q\mathbb{Z})$. Show that $\sum_{n\in\mathbb{Z}} c(n)\varphi(n) = \sum_{k\in\mathbb{Z}} \hat{c}(-k)\hat{\varphi}(k/q)$.
   (b) Let χ be a primitive Dirichlet character mod q. Show that
   $$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \frac{G(\chi)}{q} \sum_{k\in\mathbb{Z}} \bar{\chi}(k)\hat{\varphi}\left(\frac{k}{q}\right).$$

9. Combine the Vinogradov trick and the Burgress bound.

Application: Weyl differencing and equidistribution on the circle

10. (Equidistribution) Let X be a compact space, μ a fixed probability measure on X (thought of as the "uniform" measure). We say that a sequence of probability measures $\{\mu_n\}_{n=1}^{\infty}$ is *equidistributed* if it converges to μ in the weak-* sense, that is if for every $f \in C(X)$, $\lim_{n\to\infty} \mu_n(f) = \mu(f)$ (equivalently, if for every open set $U \subset X$, $\mu_n(U) \to \mu(U)$).
   (a) Show that it is enough to check convergence on a set $B \subset C(X)$ such that $\text{Span}_{\mathbb{C}}(B)$ is dense in $C(X)$.
   (b) (Weyl criterion) We will concentrate on the case $X = \mathbb{R}/\mathbb{Z}$, $\mu =$ Lebesgue. Show that in that case it is enough to check whether $\int_0^1 e(kx)\, \mathrm{d}\mu_n(x) \xrightarrow[n\to\infty]{} 0$ for each non-zero $k \in \mathbb{Z}$. (Hint: Stone–Weierstrass)
   DEF We say that a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is equidistributed (w.r.t. μ) if the sequence $\left\{\frac{1}{n}\sum_{k=1}^{n} \delta_{x_k}\right\}_{k=1}^{\infty}$ is equidistributed, that is if for every open set U the proportion of $1 \le k \le n$ such that $x_k \in U$ converges to $\mu(U)$, the proportion of the mass of X carried by μ.
   (c) Let α be irrational. Show directly that the sequence fractional parts $\{n\alpha \mod 1\}_{n=1}^{\infty}$ is dense in $[0,1]$.
   (d) Let α be irrational. Show that the sequence of fractional parts $\{n\alpha \mod 1\}_{n=1}^{\infty}$ is equidistributed in $[0,1]$.
   (e) Returning to the setting of parts (a),(b). suppose that $\text{supp}(\mu) = X$. Show that every equidistributed sequence is dense.