Math 539: Problem Set 0 (due 18/1/2016)

Policy for Problem Sets: adopt a reasonable workload for your situation; you should do some non-trivial problems in any case.

In this problem set: We will repeatedly rely on this results of this problem set; there is no number theory here yet. If you only do some of the problems, I recommend starting with 1,3 and parts of of 5 and 6. Problem 7 is intended to review ideas from Math 437/537.

Real analysis

1. (Asymtotic notation) Let f,g be defined for x large enough. We write $f \ll g$ and $f = O(g)$ if there is $C > 0$ such that $|f(x)| \le Cg(x)$ for all large enough x.
   (a) Let f,g be functions such that $f(x), g(x) > 2$ for x large enough. Show that $f \ll g$ implies $\log f \ll \log g$. Give a counterexample under the weaker hypothesis $f(x), g(x) > 1$.
   (b) For all $A > 0$, $0 < b < 1$ and $\varepsilon > 0$ show that for $x \ge 1$,
   $$\log^A x \ll \exp\left(\log^b x\right) \ll x^{\varepsilon}.$$
2. Set $\log_1 x = \log x$ and for x large enough, $\log_{k+1} x = \log\left(\log_k x\right)$. Fix $\varepsilon > 0$.
   (PRAC) Find the interval of definition of $\log_k x$. For the rest of the problem we suppose that $\log_k x$ is defined at N.
   (a) Show that $\sum_{n=N}^{\infty} \frac{1}{n \log n \log_2 n \cdots \log_{k-1} n (\log_k n)^{1+\varepsilon}}$ converges.
   (b) Show that $\sum_{n=N}^{\infty} \frac{1}{n \log n \log_2 n \cdots \log_{k-1} n (\log_k n)^{1-\varepsilon}}$ diverges.
3. (Stirling's formula)
   (a) Show that $\int_{k-1/2}^{k+1/2} \log t \, dt - \log k = O(\frac{1}{k^2})$.
   (b) Show that there is a constant C such that
   $$\log(n!) = \sum_{k=1}^{n} \log k = \left(n + \frac{1}{2}\right) \log n - n + C + O\left(\frac{1}{n}\right).$$
   RMK $C = \frac{1}{2}\log(2\pi)$ (see problem 6(f) below) but this is rarely relevant.
4. Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be sequences with partial sums $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$.
   (a) (Abel summation formula) $\sum_{n=1}^{N} a_n b_n = A_N b_N - \sum_{n=1}^{N-1} A_n (b_{n+1} - b_n)$
   – (Summation by parts formula) Show that $\sum_{n=1}^{N} a_n B_n = A_N B_N - \sum_{n=1}^{N-1} A_n b_{n+1}$.
   (b) (Dirichlet's test) Suppose that $\{A_n\}_{n=1}^{\infty}$ are uniformly bounded and that $b_n \in \mathbb{R}_{>0}$ decrease monotonically to zero. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.
   (c) Let $\chi_3(n) = \begin{cases} \pm 1 & n \equiv \pm 1\,(3) \\ 0 & 3|n \end{cases}$. Show that *Dirichlet's L-series* $L(s;\chi_3) = \sum_{n=1}^{\infty} \chi_3(n) n^{-s}$ coverges for s real and positive.

Complex analysis: the Gamma function

DEFINITION. The *Mellin transform* of a function ϕ on $(0,\infty)$ is given by $\mathcal{M}\phi(s) = \int_0^\infty \phi(x)x^s \frac{\mathrm{d}x}{x}$ whenever the integral converges absolutely.

5. Let ϕ be a bounded function on $(0,\infty)$ [measurable so the integrals make sense]
   (a) Suppose that for some $\alpha > 0$ we have $\phi(x) = O(x^{-\alpha})$ as $x \to \infty$ (see problem 1 for this notation). Show that the $\mathcal{M}\phi$ defines a holomorphic function in the strip $0 < \Re(s) < \alpha$.
   For the rest of the problem assume that $\phi(x) = O(x^{-\alpha})$ holds for *all* $\alpha > 0$.
   (b) Suppose that ϕ is smooth in some interval $[0,b]$ (that is, there $b > 0$ and is a function $\psi \in C^\infty([0,b])$ such that $\psi(x) = \phi(x)$ with $0 < x \leq b$). Show that $\tilde{\phi}(s)$ extends to a meromorphic function in $\Re(s) < \alpha$, with at most simple poles at $-m$, $m \in \mathbb{Z}_{\geq 0}$ where the residues are $\frac{\phi^{(m)}(0)}{m!}$ (in particular, if this derivative vanishes there is no pole).
   (c) Extend the result of (b) to ϕ such that $\phi(x) - \sum_{i=1}^r \frac{a_i}{x^i}$ is smooth in an interval $[0,b]$.
   (d) Let $\Gamma(s) = \int_0^\infty e^{-t}t^s \frac{\mathrm{d}t}{t}$. Show that $\Gamma(s)$ extends to a meromorphic function in $\mathbb{C}$ with simple poles at $\mathbb{Z}_{\leq 0}$ where the residue at $-m$ is $\frac{(-1)^m}{m!}$.

6. (The Gamma function) Let $\Gamma(s) = \int_0^\infty e^{-t}t^s \frac{\mathrm{d}t}{t}$, defined initially for $\Re(s) > 0$. See supplementary problem B for a proof that this extends to a meromorphic function in $\mathbb{C}$ and a determination of the location and residues at the poles (all poles are simple).
   FACT A standard integration by parts shows that $s\Gamma(s) = \Gamma(s+1)$ and hence $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}_{\geq 1}$.
   (a) Let $Q_N(s) = \int_0^N \left(1 - \frac{x}{N}\right)^N x^s \frac{\mathrm{d}x}{x}$. Show that $Q_N(s) = \frac{N!}{s(s+1)\cdots(s+N)}N^s$. Show that $0 \leq \left(1 - \frac{x}{N}\right)^N \leq e^{-x}$ holds for $0 \leq x \leq N$, and conclude that $\lim_{N\to\infty} \frac{N!}{s(s+1)\cdots(s+N)}N^s = \Gamma(s)$ for on $\Re s > 0$ (for a quantitative argument show instead $0 \leq e^{-x} - \left(1 - \frac{x}{N}\right)^N \leq \frac{x^2}{N}e^{-x}$)
   (b) Define $f(s) = se^{\gamma s}\prod_{n=1}^\infty \left(1 + \frac{s}{n}\right)e^{-s/n}$ where $\gamma = \lim_{n\to\infty}\left(\sum_{i=1}^n \frac{1}{i} - \log n\right)$ is Euler's constant. Show that the product converges locally uniformly absolutely and hence defines an entire function in the complex plane, with zeros at $\mathbb{Z}_{\leq 0}$. Show that $f(s+1) = \frac{1}{s}f(s)$.
   (c) Let $P_N(s) = se^{\gamma s}\prod_{n=1}^N \left(1 + \frac{s}{n}\right)e^{-s/n}$. Show that for $\alpha \in (0,\infty)$,$\lim_{N\to\infty} Q_N(\alpha)P_N(\alpha) = 1$ and conclude (without using problem 5!) that $\Gamma(s)$ extends to a meromorphic function in $\mathbb{C}$ with simple poles at $\mathbb{Z}_{\leq 0}$, that $\Gamma(s) \neq 0$ for all $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and that it has the Weierstraß product representation

$$\Gamma(s) = \frac{e^{-\gamma s}}{s}\prod_{n=1}^\infty \left(1 + \frac{s}{n}\right)^{-1} e^{s/n}.$$

   (d) Let $\digamma(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ be the Digamma function. Using the Euler–Maclaurin summation formula $\sum_{n=0}^{n=N} f(n) = \int_0^N f(x)\,\mathrm{d}x + \frac{1}{2}(f(0) + f(N)) + \frac{1}{12}(f'(0) - f'(N)) + R$, with $|R| \leq \frac{1}{12}\int_0^N |f''(x)|\,\mathrm{d}x$, show that if $-\pi + \delta \leq \arg(s) \leq \pi + \delta$ and s is non-zero then

$$\digamma(s) = \log s - \frac{1}{2s} + O_\delta\left(|s|^{-2}\right).$$

Integrating on an appropriate contour, obtain *Stirling's Approximation*: there is a constant c such that if $\arg(s)$ is as above then

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + c + O_\delta\left(\frac{1}{|s|}\right).$$

RMK Compare with the result of 3(b)

(e) Show *Euler's reflection formula*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Conclude that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and hence that $\int_{-\infty}^{+\infty} e^{-\alpha x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$.

(f) Setting $s = \frac{1}{2} + it$ in the reflection formula and letting $t \to \infty$, show that $c = \frac{1}{2}\log(2\pi)$ in Stirling's Approximation.

(g) Show *Legendre's duplication formula*

$$\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = \sqrt{\pi}2^{1-s}\Gamma(s).$$

Review of arithmetic functions

REMARK We won't do any serious abstract algebra in this course, but I will use basic terminology like "commutative ring". For definitions Wikipedia is your friend.

7. Most of the stuff below should be familiar from Math 437/537

DEF (Dirichlet convolution) $(f * g)(n) = \sum_{ab=n} f(a)g(b)$.

(a) The set of arithmetic functions with pointwise addition and Dirichlet convolution forms a commutative ring. The identity element is the function $\delta(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$.

(b) f is invertible in this ring iff $f(1)$ is invertible in $\mathbb{C}$.

(c) If f, g are multiplicative so is $f * g$.

DEF $I(n) = 1$, $N(n) = n$, $\phi(n) = \left|(\mathbb{Z}/n\mathbb{Z})^\times\right|$, $\mu(n) = (-1)^r$ if n is a product of $r \geq 0$ distinct primes, $\mu(n) = 0$ otherwise (i.e. if n is divisible by some p^2).

(d) ("Möbius inversion") Show that $I * \mu = \delta$ by explicitly evaluating the convolution at $n = p^m$ and using (c).

(e) Show that $\phi * I = N$: (i) by explcitly evaluating the convolution at $n = p^m$ and using (c); (ii) by a combinatorial argument.

DEF A *derivation* in the ring R is a function $D\colon R \to R$ such that for all $f, g \in R$ one has $D(fg) = Df \cdot g + f \cdot Dg$ (example: $D = \frac{\mathrm{d}}{\mathrm{d}x}$ acting on smooth functions).

(f) Show that pointwise multiplication by an arithmetic function $L(n)$ is a derivation in the ring of arithmetic functions iff $L(n)$ is completely additive: $L(de) = L(d) + (e)$ for all $d, e \geq 1$. In particular, this applies to $L(n) = \log n$.

Supplement: Formal Dirichlet Series

Supplementary problems are not for submission.

DEFINITION. The *ring of Dirichlet polynomials* (let's denote it $\mathcal{D}_{\mathrm{f}}$) consists of all formal expressions of the form $D(s) = \sum_{n \le x} a_n n^{-s}$ where $a_n \in \mathbb{C}$ (modulu the obvious equivalence relation). Call a_1 the *constant coefficient*. Multiplication is the bilinear map induced from $n^{-s} \times m^{-s} = (nm)^{-s}$.

A. (Basics)
   (a) Show that this is a ring, and that the map $D(s) \mapsto a_1$ is a ring homomorphism $Q \colon \mathcal{D}_{\mathrm{f}} \to \mathbb{C}$. Write $\mathfrak{m}$ for its kernel, the maximal ideal.
   (b) Conversely, show that every homomorphism $\mathcal{D}_{\mathfrak{f}} \to \mathbb{C}$ is of the form $\sigma \circ Q$ for some $q \in \mathrm{Aut}(\mathbb{C})$.
   (b) Let $v \colon \mathcal{D} \colon [0, \infty]$ be given by $v(\sum_n a_n n^{-s}) = N$ if $a_N \neq 0$ but $a_n = 0$ for $n < N$ (set $v(0) = \infty$). Show that $v(D_1 + D_2) \geq \min\{v(D_1), v(D_2)\}$ and conclude that $d(D_1, D_2) = \exp\{-v(D_1 - D_2)\}$ is a metric on $\mathcal{D}_{\mathrm{f}}$ (in fact, an ultrametric).
   (c) Show that the ring $\mathcal{D}$ of Dirichlet series is exactly the completion of $\mathcal{D}_{\mathrm{f}}$ with respect to the metric.
   (d) Show that for any arithmetic function f, the series $\sum_{n \ge 1} f(n) n^{-s}$ (thought of as a sum of the individual Dirichlet polynomials $f(n) n^{-s}$) converges in $\mathcal{D}$ to formal series $D_f(s) = \sum_{n \ge 1} f(n) n^{-s}$.
   (e) (calculus student's dream) Show that (for $D_i \in cD$) a series $\sum_i D_i$ converges iff the terms converge to zero (i.e. iff $v(D_i) \to \infty$).
   (f) Show that the product $\prod_i (1 + D_i)$ converges and diverges under the same hypothesis.

B. (exp and log)
   (a) Let $F(T) \in T\mathbb{C}[[T]]$ be a formal power series with no constant coefficient, say $F(T) = \sum_{k=1}^{\infty} a_k T^k$, and let $D \in \mathfrak{m}$ be a formal Dirichlet series with no constant coefficient. Show that $F(D) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} a_k D^k$ converges in our topology to an element of $frakm$, so that $F \colon \mathfrak{m} \to \mathfrak{m}$ is continous.
   (b) Show that same for a two-variable power series with no constant coefficient, $G(T, S) \in (T + S)\mathbb{C}[[T, S]]$.
   (c) Conclude that $\log(1 + D)$, $\exp(D)$ exist for $D \in \mathfrak{m}$ and satisfy $\log((1 + D_1)(1 + D_2)) = \log(1 + D_1) + \log(1 + D_2)$ and $\exp(D_1 + D_2) = \exp(D_1)\exp(D_2)$.
   (d) Show that the construction above respects composition of formal power series with no constant coefficient, and conclude that $\exp\log D = D$ and that $\log\exp D = D$.
   (e) Extend exp to all of $\mathcal{D}$ using the topology of pointwise convergence of the coefficients.
   (f) The *formal derivative* of $D(s) = \sum_{n \ge 1} f(n) n^{-s} \in \mathcal{D}$ is the series $D'(s) = \sum_{n \ge 1} (f(n) \log n) n^{-s}$. In 7(f) you obtained the Leibnitz identity $(D_1 D_2)' = D_1' D_2 + D_1 D_2'$. Show that $(\log D)' = \frac{D'}{D}$ and that $(\exp D)' = (\exp D) D'$.

C. (Euler products)
   (a) For each prime p let D_p be a formal Dirichlet series supported on the powers of p, with constant coefficient 1. Show that $\prod_p D_p$ converges in cD. Series obtained this way are said to have an *Euler product*.
   (b) Show that every series has at most one representation as an Euler product, and that if D_1, D_2 have an Euler product then so does $D_1 D_2$.