Lecture 18: Applications of Sylow Thms

Problem: Let G be a finite group, $M < G$ of index p, where p is the smallest prime dividing $\#G$. Then $M \triangleleft G$.

Solution: Let G act on $X = G/M$ by translation.

Get homomorphism $f: G \to S_X \simeq S_p$ $\quad (\#X = [G:M] = p)$

Note: $(\#G, \#S_p) = p$, try to make subgp of S_p of order dividing $\#G$.

Let $H = f(G) = \mathrm{Im}(G) < S_p$.

(1) $\#H \mid \#G$ because by 1st isom thm, $H \simeq G/\mathrm{Ker}(f)$ so $\#H = [G:\mathrm{Ker}(f)] \mid \#G$

(2) $\#H \mid p! = \#S_p$ by Lagrange

but $p! = (p-1)! \cdot p$ and $(p-1)!$ is prime to $\#G$.

so $\#H \mid p$, so $\#H$ is either 1 or p.

But G acts transitively on G/M, so H acts transitively, so $\#H = p$

[or if $g \in G, g \notin M$ then $gM \neq M$ so $f(g) \neq id_X$ so $H \neq \{id\}$.]

By 1st isom thm, if $\mathrm{Im}(f)$ has order p, $[G:\mathrm{Ker}(f)] = p$.

But $[G:M] = p$. Now if $g \notin M$ then $g \notin \mathrm{Ker} f$, so

$$\mathrm{Ker}(f) \subset M.$$

so $[M:\mathrm{Ker}(f)] = \dfrac{[G:\mathrm{Ker}(f)]}{[G:M]} = 1$ and $M = \mathrm{Ker}(f)$ is normal

Classification of groups of order 12

Recap: let G be a finite group of order $n = p^r \cdot m$, p prime, $p \nmid m$, $r \geq 1$

Then: (1) G has subgps of order p^r.

(2) They are all conjugate, their number $n_p(G) \mid m$.

(3) $n_p(G) \equiv 1 \ (p)$

Example: $n = 12 = 2^2 \cdot 3^1$. $n_2(G) \mid 3$, odd so $n_2(G) \in \{1, 3\}$

$n_3(G) \mid 4$, so $n_3(G) \in \{1, 4\}$ $\quad (2 \not\equiv 1 \ (3))$

Case 1: $n_3(G) = 4$. Then the action of G by conjugation on $Syl_3(G)$ gives a hom $f : G \to S_{Syl_3(G)} \cong S_4$.

What is $Ker(f)$? That consists of those $g \in G$ that normalize all 3-Sylow subgps. Let P_3 be a 3-Sylow subgp. Then $\# P_3 = 3$, $[G : N_G(P_3)] = 4$ ← number of conjugates

so
$$\left.\begin{array}{l} 4\left(\begin{array}{c} G \\ | \\ N_G(P_3) \end{array}\right. \\ \qquad | \\ 3\left(\begin{array}{c} P_3 \\ | \\ \{e\} \end{array}\right. \end{array}\right) 12$$

in words: $[G : P_3] = \frac{\# G}{\# P_3} = \frac{12}{3} = 4$

~~key obser~~ and $P_3 \subset N_G(P_3)$

so $P_3 = N_G(P_3)$

~~Now~~ So $Ker(f) = \bigcap Syl_3(G) =$ intersection of the 3-Sylow subgps $= \{e\}$

(intersection of distinct subgps $\cong C_3$ is a proper subgp of at least one, hence trivial)

Conclusion: $G \cong$ subgp of S_4, of order 12.

Want to show $G \cong A_4$. For this note: if $g \in P_3 \setminus \{e\}$, g has order 3, so $f(g)$ has order 3, so $f(g)$ is a 3-cycle.

G has 8 elements of order 3: 4 3-sylow subgps each has $3-1=2$ elements of order 3 (these are all distinct because we saw the subgps are disjoint)

S_4 has $\frac{4\cdot 3\cdot 2}{3}=8$ 3-cycles : 3-cycle has form (ijk)
$=(jki)=(kij)$

So $f(G)$ contains all 3-cycles, hence the subgp they generate, which is A_4, so $f(G)\supseteq A_4$, but $\#f(G)=12=\#A_4$, so $f(G)=A_4$ and $G\simeq A_4$.

<u>Alternative</u>: Show that A_4 is the only subgp of S_4 of order 12:
let $H<S_4$ have order 12. By Cauchy, H contains σ of order 3, which is a 3-cycle (other cycle structures are $()$, (12), $(12)(34)$, (1234))
But H is <u>normal</u> ($[S_4:H]=2$), so H contains all conjugates of σ.
So $H\supset \langle\{\text{3-cycles}\}\rangle = A_4$, so $H=A_4$.

(PS: if $[G:H]=2$, H is normal)
(PS: if $[G:H]=p$, p smallest prime $\mid \#G$, then H is normal)

<u>Alternative</u>: $f(G)$ contains 4 3-sylow subgps
Consider $Syl_3(S_4)$. $\#S_4=24=8\cdot 3$ so 3-sylow subgps of S_4 are of order 3, and their number $n_3(S_4)\in\{1,2,4,8\}$, $n_3(S_4)\equiv 1\ (3)$, so $n_3(S_4)\in\{1,4\}$ but $n_3(S_4)\geq n_3(f(G))$ so they are equal: $f(G)$ contains all elements of order 3 in S_4.

<u>Bottom line</u>: if $\#G=12$, $n_3(G)=4$ then $G\simeq A_4$.

Otherwise,

<u>Case 2:</u> $n_3(G) = 1$, Now the 3-sylow subgp P is normal

let Q be a 2-sylow subgp, $\# Q = 4$.

Then $G = PQ = P \rtimes Q$. ($P \cap Q = \{e\}$ because ~~gcd~~ $\gcd(\#P, \#Q) = 1$)

$\#P \cdot \#Q = \# G$

Now $P \cong C_3$. <u>Remains:</u> (1) classify Q (2) Classify actions of Q on P

(1) Q is isomorphic to one of C_4, $C_2 \times C_2$

(2) <u>Case 2a:</u> $Q \cong C_4 = \{1, b, b^2, b^3\}$

need action $Q \to \mathrm{Aut}(P) = \{+, -\}$.

two homs: either $\varphi(b) = +$, φ triv, get $C_4 \times C_3$

or $\varphi(b) = -$, then $\varphi(b^2) = +$, $\varphi(b^3) = -$.

(this is the map $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ given by "reduction mod 2")

get a non-commutative pdt $C_4 \ltimes C_3$.

<u>Case 2b:</u> $Q \cong C_2 \times C_2$.

Need to classify $\mathrm{Hom}(\underset{Q}{C_2 \times C_2}, \underset{\mathrm{Aut}(P)}{C_2})$

either ~~image~~ image is <u>trivial</u>, then $G \cong (C_2 \times C_2) \times C_3$

<u>or:</u> choose ~~a~~ $a_1 \in Q$ ~~with~~ with $\varphi(a_1) \neq id_P$

choose $a_2 \in \mathrm{Ker}(\varphi)$, $a_2 \neq e$.

<u>Claim:</u> $Q = \{1, a_1, a_2, a_1 a_2\}$

so $Q \cong C_2 \times C_2$, where 1^{st} copy of C_2 acts on C_3

2^{nd} doesn't. so $Q \ltimes C_3 \cong (C_2 \times C_2) \ltimes C_3 \cong C_2 \times (C_2 \ltimes S_3)$

$\cong C_2 \times S_3$

$\varphi \in \mathrm{Hom}(F^2, F)$

~~choose~~ $\varphi \neq 0$

let $\underline{u} \in \mathrm{Ker}(\varphi) \neq \underline{0}$

$\underline{v} \in F^2 : \varphi(\underline{v}) \neq 0$.

Then $\{\underline{u}, \underline{v}\}$ is a basis

Conclusion: Gps of order 12, up to isom are:

$$A_4, \quad C_3 \times C_4, \quad C_3 \rtimes C_4, \quad C_2 \times C_2 \times C_3, \quad C_2 \times S_3$$

C_{12}