

Math 322: Problem Set 8 (due 6/11/2015)

On group actions and homomorphisms

1. Let the group G act on the set X .
DEF The *kernel* of the action is the normal subgroup $K = \{g \in G \mid \forall x \in X : g \cdot x = x\}$.
PRAC K is the kernel of the associated homomorphism $G \rightarrow S_X$, hence $K \triangleleft G$ indeed.
(a) Construct an action of G/K on X “induced” from the action of G .
DEF An action is called *faithful* if its kernel is trivial.
(b) Show that the action of G/K on X is faithful.
SUPP Show that this realizes G/K as a subgroup of S_X .
(c) Suppose G acts non-trivially on a set of size n . Show that G has a proper normal subgroup of index at most $n!$.
(*d) Show that an infinite simple group has no proper subgroups of finite index.
- *2. Let G be a group of finite order n , and let p be the smallest prime divisor of n . Let $M < G$ be a subgroup of index p . Show that M is normal.
RMK In particular, this applies when G is a finite p -group.

Automorphisms of groups and semidirect products

Recall that $\text{Aut}(H)$ is the group of isomorphisms $H \rightarrow H$.

- *3. Let H, N be groups, and let $\varphi \in \text{Hom}(H, \text{Aut}(N))$ be an action of H on N by automorphisms. We write φ_h rather than $\varphi(h)$ for the automorphism given by $h \in H$, so result of h acting on n (the result of applying the automorphism $\varphi(h)$ to n) will be written $\varphi_h(n)$. That φ is a homomorphism is the statement that $\varphi_h \circ \varphi_{h'} = \varphi_{hh'}$.
DEF The (external) *semidirect product* of H and N along φ is the operation

$$(h_1, n_1) \cdot (h_2, n_2) = \left(h_1 h_2, \left(\varphi_{h_2^{-1}}(n_1) \right) n_2 \right)$$

on the set $H \times N$. We denote this group $H \rtimes_{\varphi} N$.

- PRAC Verify that when φ is the trivial homomorphism ($\varphi_h = \text{id}$ for all $h \in H$), this is the ordinary direct product.
- (a) Show that the semidirect product is, indeed, a group.
 - (b) Show that $f_H: H \rightarrow H \rtimes_{\varphi} N$ given by $f(h) = (h, e)$, $f_N: N \rightarrow H \rtimes_{\varphi} N$ given by $f(n) = (e, n)$ and $\pi: H \rtimes_{\varphi} N \rightarrow H$ given by $\pi(h, n) = h$ are group homomorphisms.
 - (c) Show that $\tilde{H} = f_H(H)$ and $\tilde{N} = f_N(N)$ are subgroups with \tilde{N} normal. Show that for $\tilde{h} = (h, e)$ and $\tilde{n} = (e, n)$ we have $\tilde{h}\tilde{n}\tilde{h}^{-1} = (\varphi(h)(n))$.
 - (d) Show that $H \rtimes_{\varphi} N$ is the internal semidirect product of its subgroups \tilde{H}, \tilde{N} .

4. (Concrete 3(b),(c),(d)) Let $H = \mathbb{R}^\times$ act on $N = \mathbb{R}$ by multiplication (so $\varphi_h(n) = hn$). Show $H \rtimes_\varphi N$ is isomorphic to the subgroup $P = \left\{ \begin{pmatrix} h & n \\ & 1 \end{pmatrix} \mid h \in \mathbb{R}^\times, n \in \mathbb{R} \right\}$ of $\text{GL}_2(\mathbb{R})$.
 SUPP Do the same with $H = (\mathbb{Z}/n\mathbb{Z})^\times$, $N = \mathbb{Z}/n\mathbb{Z}$. Now P is a finite group.
 SUPP Same with $H = \text{GL}_d(\mathbb{R})$, $N = \mathbb{R}^d$, $P = \left\{ \begin{pmatrix} h & \underline{n} \\ & 1 \end{pmatrix} \mid h \in \text{GL}_d(\mathbb{R}), \underline{n} \in \mathbb{R}^d \right\} < \text{GL}_{d+1}(\mathbb{R})$.
5. (Cyclic groups)
- (a) Let A be a group. Show that mapping $f \in \text{Hom}(C_n, A)$ to $f([1]_n)$ gives a bijection between $\text{Hom}(C_n, A)$ and the set of $a \in A$ of order dividing n .
- (b) Write f_a for the homomorphism such that $f([1]_n) = a$. When $A = C_n = (\mathbb{Z}/n\mathbb{Z}, +)$ show that $f_a \circ f_b = f_{ab}$ (ab is multiplication mod n) and hence that $\text{Aut}(C_n) \simeq (\mathbb{Z}/n\mathbb{Z})^\times$.
- RMK You've just done a fancy version of problem 4 of PS1

Bonus problem

6. The two parts complete problem 3. For these let $\varphi \in \text{Hom}(H, \text{Aut}(N))$.
- (a) For $\alpha \in \text{Aut}(H)$ define $\psi: H \rightarrow \text{Aut}(N)$ by $\psi = \varphi \circ \alpha$ (that is $\psi_h = \varphi_{\alpha(h)}$). Show that $F(h, n) = (\alpha^{-1}(h), n)$ gives an isomorphism $F: H \rtimes_\varphi N \rightarrow H \rtimes_\psi N$.
- (b) For $\beta \in \text{Aut}(N)$ define $\psi: H \rightarrow \text{Aut}(N)$ by $\psi_h = \beta \circ \varphi_h \circ \beta^{-1}$ (this is conjugation in $\text{Aut}(N)$!). Show that $H \rtimes_\varphi N \simeq H \rtimes_\psi N$.
- (c) Let $a, b \in \text{Aut}(N)$ generate the same cyclic subgroup, and let $f_a, f_b \in \text{Hom}(C_n, \text{Aut}(N))$ be the maps from 5(b). Show that $C_n \rtimes_{f_a} N \simeq C_n \rtimes_{f_b} N$
- RMK From (b),(c) we conclude and conclude that semidirect products $C_n \rtimes N$ are determined by *conjugacy classes of subgroups* of $\text{Aut}(N)$ of order dividing n .

Supplementary problems

- A. We show that $(\mathbb{Z}/p\mathbb{Z})^\times \simeq C_{p-1}$ so that $\text{Aut}(C_p) \simeq C_{p-1}$.
- Let F be a field. Show that F^\times has at most d elements of order dividing d (hint: a polynomial of degree d over a field has at most d roots).
 - Let $H < F^\times$ be a finite group. Show that H is cyclic.
 - Show that $\text{Aut}(C_p) \simeq C_{p-1}$.

Solving the following problem involves many parts of the course.

- B. Let G be a group of order 8.
- Suppose G is commutative. Show that G is isomorphic to one of $C_8, C_4 \times C_2, C_2 \times C_2 \times C_2$.
 - Suppose G is non-commutative. Show that there is $a \in G$ of order 4 and let $H = \langle a \rangle$.
 - Show that $a \notin Z(G)$ but $a^2 \in Z(G)$.
 - Suppose there is $b \in G - H$ of order 2. Show that $G \simeq D_8$ (hint: $bab^{-1} \in \{a, a^3\}$ but can't be a).
 - Let $b \in G - H$ have order 4. Show that $bab^{-1} = a^3$ and that $a^2 = b^2 = (ab)^2$.
 - Setting $c = ab, -1 = a^2$ and $-g = (-1)g$ show that $G = \{\pm 1, \pm a, \pm b, \pm c\}$ with the multiplication rule $ab = c, ba = -c, bc = a, cb = -a, ca = b, ac = -b$.
 - Show that the set in (f) with the indicated operation is indeed a group.
- DEF The group of (f),(g) is called the *quaternions* and indicated by Q .

- C. Let G be a group (especially infinite).
- DEF Let X be a set. A *chain* $\mathcal{C} \subset P(X)$ is a set of subsets of X such that if $A, B \in \mathcal{C}$ then either $A \subset B$ or $B \subset A$.
- Show that if \mathcal{C} is a chain then for every finite subset $\{A_i\}_{i=1}^n \subset \mathcal{C}$ there is $B \in \mathcal{C}$ such that $A_i \subset B$ for all i .
 - Suppose \mathcal{C} is a non-empty chain of subgroups of a group G . Show that the union $\bigcup \mathcal{C}$ is a subgroup of G containing all $A \in \mathcal{C}$.
 - Suppose \mathcal{C} is a chain of p -subgroups of G . Show that $\bigcup \mathcal{C}$ is a p -group as well.
 - (*d) Use Zorn's Lemma to show that every group has maximal p -subgroups (p -subgroups which are not properly contained in other p -subgroups), in fact that every p -subgroup is contained in a maximal one.

RMK When G is infinite, it does not follow that these maximal subgroups are all conjugate.