Math 101 – SOLUTIONS TO WORKSHEET 24
SERIES

1. Skill 1: Geometric series and decimal expansions

(1) (Final 2013) Find the sum of the series $\sum_{n=2}^{\infty} \frac{3\cdot 4^{n+1}}{8\cdot 5^n}$. Simplify your answer.

Solution: We write this as $\sum_{n=2}^{\infty} \frac{12}{8}\left(\frac{4}{5}\right)^n$ so this is a geometric series with ratio $\frac{4}{5}$ and first term $\frac{3}{2}\left(\frac{4}{5}\right)^2$. Its sum is therefore

$$\frac{3}{2}\frac{(4/5)^2}{1-\frac{4}{5}} = \frac{3\cdot 16}{2\cdot 5\cdot 5\cdot(1-\frac{4}{5})} = \frac{24}{5\cdot(5-4)} = \boxed{\frac{24}{5}}.$$

(2) Express each decimal expansion using a geometric series, sum the series, then simplify to obtain a rational number.

(a) $0.333333\ldots$

Solution: We have $0.33333\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = \frac{3}{10}\cdot\frac{1}{1-\frac{1}{10}} = \frac{3}{9} = \boxed{\frac{1}{3}}$.

(b) $0.5757575757\ldots$

Solution: This is $\frac{57}{100} + \frac{57}{(100)^2} + \frac{57}{(100)^3} + \cdots = \frac{57}{100}\cdot\frac{1}{1-\frac{1}{100}} = \frac{57}{99}$.

(c) $0.6545454545454\ldots$

Solution: Here we have to be more careful:

$$\begin{aligned} 0.6545454545454\ldots &= 0.6 + \frac{54}{1000} + \frac{54}{100,000} + \frac{54}{10,000,000} + \cdots = 0.6 + \frac{54}{1000}\left(1 + \frac{1}{100} + \frac{1}{(100)^2} + \frac{1}{(100)^3} + \cdots\right) \\ &= 0.6 + \frac{54}{1000}\cdot\frac{1}{1-\frac{1}{100}} = \frac{6}{10} + \frac{54}{10\cdot 99} = \frac{3}{5} + \frac{3}{5\cdot 11} = \frac{3\cdot 12}{5\cdot 11} = \frac{36}{55}. \end{aligned}$$

2. Skill 2: Telecoping series

(3) Write an expression for the partial sums, decide if the series converges, and if so determine the sum.

(a) $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

Solution: We have $\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$ (partial fractions). Writing the partial sum

$$\left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{4}-\frac{1}{6}\right) + \cdots + \left(\frac{1}{n-1}-\frac{1}{n+1}\right) + \left(\frac{1}{n}-\frac{1}{n+2}\right)$$

we see that every fraction appears twice (with opposite signs) except for $1, \frac{1}{2}, -\frac{1}{n+1}, -\frac{1}{n+2}$ so

$$s_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}.$$

Thus

$$\lim_{n\to\infty} s_n = \frac{3}{2} - 0 - 0 = \frac{3}{2}$$

and the series converges.

(b) $\sum_{n=0}^{\infty} (\tan(n) - \tan(n+1))$

Solution: The function oscillates and the sequence is divergent.

(c) $\sum_{n=1}^{\infty} \left(n^2 - (n+1)^2\right)$

Solution: The nth partial sum is $\left(1^2-2^2\right) + \left(2^2-3^2\right) + \cdots + \left(n^2-(n+1)^2\right) = 1^2 - (n+1)^2$ and these clearly tend to $-\infty$ as $n \to \infty$ so the series diverges.

Date: 7/3/2016, Worksheet by Lior Silberman.