Math 101 – SOLUTIONS TO WORKSHEET 19
IMPROPER INTEGRALS

1. Improper at infinity

(1) For which values of p does $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ converge? Diverge?

Solution: For $p \neq -1$ we have $\int_1^T \frac{1}{x^p}\,\mathrm{d}x = \left[\frac{1}{1-p}x^{1-p}\right]_1^T = \frac{1}{1-p}\left(T^{1-p}-1\right)$. If $1-p>0$ then $T^{1-p}\to\infty$ so the integral diverges. If $1-p<0$ then $T^{1-p}\to 0$ and the integral converges. If $1-p=0$ then $\int_1^T \frac{1}{x^p}\,\mathrm{d}x = [\log x]_1^T = \log T \to \infty$ and the integral diverges.

(2) (Final, 2010) Evaluate $\int_{-\infty}^{-1} e^{2x}\,\mathrm{d}x$. Simplify your answer as much as possible.

Solution: $\int_T^{-1} e^{2x}\,\mathrm{d}x = \left[\frac{1}{2}e^{2x}\right]_{x=T}^{x=-1} = \frac{1}{2}\left(e^{-2}-e^{T}\right)$. Now $\lim_{T\to-\infty} e^{2T} = \lim_{x\to-\infty} e^x = 0$ so

$$\int_{-\infty}^{-1} e^{2x}\,\mathrm{d}x = \lim_{T\to-\infty}\frac{1}{2}\left(e^{-2}-e^{2T}\right) = \boxed{\frac{1}{2}e^{-2}}.$$

Solution: $\int_{-T}^{-1} e^{2x}\,\mathrm{d}x = \left[\frac{1}{2}e^{2x}\right]_{x=-T}^{x=-1} = \frac{1}{2}\left(e^{-2}-e^{-T}\right)$. Now $\lim_{T\to\infty} e^{-2T} = \lim_{x\to\infty} e^{-x} = 0$ so

$$\int_{-\infty}^{-1} e^{2x}\,\mathrm{d}x = \lim_{T\to\infty}\frac{1}{2}\left(e^{-2}-e^{-2T}\right) = \boxed{\frac{1}{2}e^{-2}}.$$

(3) Find a constant C such that $\int_{-\infty}^{+\infty}\frac{C\,\mathrm{d}x}{1+x^2} = 1$.

Solution: $\int_0^T \frac{C\,\mathrm{d}x}{1+x^2} = C\left[\arctan x\right]_{x=0}^{x=T} = C\arctan T$ so $\int_0^\infty \frac{\mathrm{d}x}{1+x^2} = C\lim_{T\to\infty}\arctan T = \frac{\pi}{2}C$. Since the function is symmetric we also have $\int_{-\infty}^{0}\frac{C\,\mathrm{d}x}{1+x^2} = \frac{\pi}{2}C$so

$$\int_{-\infty}^{+\infty}\frac{C\,\mathrm{d}x}{1+x^2} = \pi C$$

and we need to take $\boxed{C = \frac{1}{\pi}}$.

(4) We study $\int_{-\infty}^{+\infty} x\,\mathrm{d}x$.

(a) Evaluate $\int_{-T}^{T} x\,\mathrm{d}x$.

(b) Evaluate $\lim_{T\to\infty}\int_{-T}^{T} x\,\mathrm{d}x$.

(c) Does the integral converge?

Solution: $\int_T^{-T} x\,\mathrm{d}x = 0$ since the integrand is odd, so $\lim_{T\to\infty}\int_{-T}^{T} x\,\mathrm{d}x = 0$. Nevertheless the integral diverges: $\int_0^\infty x\,\mathrm{d}x$ diverges, as does $\int_{-\infty}^0 x\,\mathrm{d}x$.

(5) (Final, 2009) For what values of p does $\int_e^\infty \frac{\mathrm{d}x}{x(\log x)^p}$ converge?

Solution: We note that $\frac{\mathrm{d}x}{x} = d(\log x)$, so letting $u = \log x$ we have $\int_{x=e}^{x=T}\frac{\mathrm{d}x}{x(\log x)^p} = \int_1^{\log T}\frac{\mathrm{d}u}{u^p}$. Now as $T\to\infty$, $\log T\to\infty$ as well, so this converges exactly when $\int_1^\infty \frac{\mathrm{d}u}{u^p}$ converges, that is exactly for $\boxed{p>1}$.

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2. Improper at finite points

(6) For which values of p does $\int_0^1 \frac{\mathrm{d}x}{x^p}$ converge?

Solution: For $p \neq 1$ we have $\int_\epsilon^1 \frac{\mathrm{d}x}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_\epsilon^1 = \frac{1}{1-p} - \frac{\epsilon^{1-p}}{1-p}$. Now as $\epsilon \to 0$,

$$\lim_{\epsilon\to 0} \epsilon^{1-p} = \begin{cases} 0 & 1-p>0 \\ \infty & 1-p<0 \end{cases}$$

so $\int_0^1 \frac{\mathrm{d}x}{x^p}$ exists when $p<1$ and diverges when $p>1$. For $p=1$ we have $\int_\epsilon^1 \frac{\mathrm{d}x}{x} = -\log\epsilon \xrightarrow[\epsilon\to 0]{} \infty$ and the integral diverges as well.

(7) (Math 103 Final, 2013) Evaluate the integral if it exists, otherwise show that it doesn't: $I = \int_0^2 \frac{\mathrm{d}x}{1-x^2}$.

Solution: The function $\frac{1}{1-x^2}$ is discountinuous at $x=1$, so we need to consider the convergence of $\int_0^1 \frac{\mathrm{d}x}{1-x^2}$ and $\int_1^2 \frac{\mathrm{d}x}{1-x^2}$ separately. Considering the first integral, for $0 \le x < 1$ we have $\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} \ge \frac{1}{1-x}\cdot\frac{1}{1+1}$. Now letting $1-x=u$ we have $\int_{x=0}^{x=1} \frac{\mathrm{d}x}{2(1-x)} = \frac{1}{2}\int_{u=0}^{u=1} \frac{\mathrm{d}u}{u}$ diverges by the p-test ($p=1$) so our integral diverges.

Solution: We have $\frac{1}{1-x^2} = \frac{1}{2}\frac{1}{1-x} + \frac{1}{2}\frac{1}{1+x}$. The second function is continuous on $[0,2]$ so it's enough to study $\frac{1}{2}\int_0^2 \frac{\mathrm{d}x}{1-x}$. Changing variables to $u=1-x$ we have

$$\int_{x=0}^{x=2} \frac{\mathrm{d}x}{1-x} = \int_{u=1}^{u=-1} \frac{-\mathrm{d}u}{u} = \int_{u=-1}^{u=1} \frac{\mathrm{d}u}{u}.$$

Now both $\int_{-1}^0 \frac{\mathrm{d}u}{u}$ and $\int_0^1 \frac{\mathrm{d}u}{u}$ diverge by the p-test so the integral diverges.

3. Comparison of integrals

(7) Decide which of the following integrals converge

(a) (103 Final, 2012) $\int_1^\infty \frac{1+\sin x}{x^2}\,\mathrm{d}x$.

Solution: Let $g(x) = \frac{1+\sin x}{x^2}$. Since $\sin x \ge -1$ for all x we have $g(x) \ge \frac{1+(-1)}{x^2} = 0$. Since $\sin x \le 1$ for all x we have $g(x) \le \frac{1+1}{x^2} = \frac{2}{x^2}$. Now $\int_1^\infty \frac{\mathrm{d}x}{x^2}$ converges by the p-test ($p=2>1$) so $\int_1^\infty \frac{2\,\mathrm{d}x}{x^2}$ converges as well. By the comparison test it follows that $\int_1^\infty g(x)\,\mathrm{d}x$ converges.

(b) $\int_1^\infty \frac{3-\cos x}{x}\,\mathrm{d}x$.

Solution: Let $f(x) = \frac{3+\cos x}{x}$. $\cos x \ge -1$ we for all x we have $f(x) \ge \frac{3+(-1)}{x} \ge \frac{1}{x}$. Now $\int_1^\infty \frac{\mathrm{d}x}{x}$ diverges by the p-test ($p=2>1$) so by the comparison test $\int_1^\infty \frac{3+\cos x}{x}\,\mathrm{d}x$ diverges as well.

(c) (Bell curve) $\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x$

Solution: By symmetric (the function is even) it's enough to consider $\int_0^\infty e^{-x^2}\,\mathrm{d}x$. Since $\int_0^1 e^{-x^2}\,\mathrm{d}x$ exists, it's enough to consider $\int_1^\infty e^{-x^2}\,\mathrm{d}x$. But for $x \ge 1$, $x^2 \ge x$ so $e^{-x^2} \le e^{-x}$. Now $\int_0^\infty e^{-x}\,\mathrm{d}x$ converges (exponential function fact) and $e^{-x^2} \ge 0$ for all x, so by the comparison test $\int_1^\infty e^{-x^2}\,\mathrm{d}x$ converges. It follows that $\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x$ converges as well.

(d) $\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}+\sin x}$

Solution: The function is continuous on $(0,1]$ and since $1<\pi$ we have $0<\sin x$ on that interval. It follows that $\sqrt{x}+\sin x > \sqrt{x} > 0$and thus $0 < \frac{1}{\sqrt{x}+\sin x} < \frac{1}{\sqrt{x}}$ on our interval. Now $\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}}$ converges by the p-test ($p=\frac{1}{2}<1$) so by the comparison test the given integral converges as well.

(e) (hard) $\int_0^1 \frac{\mathrm{d}x}{x^2+x^3}$

Solution: Multiplying the inequality $0 < x \le 1$ by x^2 we see that $x^3 < x^2$. It follows that $x^2+x^3 \le 2x^2$ so that $\frac{1}{x^2+x^3} \ge \frac{1}{2x^2}$. Now $\int_0^1 \frac{\mathrm{d}x}{2x^2} = \frac{1}{2}\int_0^1 \frac{\mathrm{d}x}{x^2}$ diverges by the $p-$test ($p=2>1$) so by the comparison test the given integral diverges as well.

(f) (hard) $\int_0^\infty \frac{x^{1000}}{e^x}\,\mathrm{d}x$

Solution: Any exponential function grows faster than any polynomial function, so that $x^{1000} \leq e^{x/2}$ for x large enough (1000 applications of l'Hôpital's rule will show $\lim_{x\to\infty} \frac{x^{1000}}{e^{x/2}} = 0$). It follows that $\frac{x^{1000}}{e^x} = \frac{x^{1000}}{e^{x/2}} \cdot \frac{1}{e^{x/2}} \leq \frac{1}{e^{x/2}}$ for x large enough. But $\int_0^\infty \frac{1}{e^{x/2}}\,\mathrm{d}x$ exists (expoential decay), so the given integral exists as well.