Math 101 – SOLUTIONS TO WORKSHEET 14
TRIGONOMETRIC SUBSTITUTION

1. More trig integrals

(1) (even power of secant) Evaluate $\int \tan^5 x \sec^4 x \, \mathrm{d}x$ using the substitution $u = \tan x$.
Solution: We have $\mathrm{d}u = \sec^2 \mathrm{d}x$ so

$$\begin{aligned}\int \tan^5 x \sec^4 x \, \mathrm{d}x &= \int \tan^5 x \sec^2 x \left(\sec^2 x \, \mathrm{d}x\right) \\ &= \int u^5 \left(1+u^2\right) \mathrm{d}u = \int \left(u^5 + u^7\right) \mathrm{d}u \\ &= \frac{1}{6}u^6 + \frac{1}{8}u^8 + C = \boxed{\frac{1}{6}\tan^6 x + \frac{1}{8}\tan^8 x + C\,.}\end{aligned}$$

(2) (odd power of tangent) Write $\int \tan^5 x \sec^3 x \, \mathrm{d}x$ in the form $\int \sin^n x \cos^m x \, \mathrm{d}x$ and evaluate it.
Solution: $\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$ so we have $\int \frac{\sin^5 x}{\cos^8 x} \mathrm{d}x$. Now the power of sine is odd, so we can set $u = \cos x \, \mathrm{d}x$, $\mathrm{d}u = -\sin x \, \mathrm{d}x$ and get

$$\begin{aligned}\int \frac{\sin^5 x}{\cos^8 x} \mathrm{d}x &= -\int \frac{\sin^4 x}{\cos^8 x} \left(-\sin x \, \mathrm{d}x\right) \\ &= -\int \frac{\left(1-u^2\right)^2}{u^8} \mathrm{d}u = -\int \frac{u^4 - 2u^2 + 1}{u^8} \mathrm{d}u \\ &= \int \left(-u^{-4} + 2u^{-6} - u^{-8}\right) \mathrm{d}u = \frac{1}{3}u^{-3} - \frac{2}{5}u^{-5} + \frac{1}{7}u^{-7} + C \\ &= \boxed{\frac{1}{3}\sec^3 x - \frac{2}{5}\sec^5 x + \frac{1}{7}\sec^7 x + C\,.}\end{aligned}$$

2. Trig substitution

(1) (Final, 2014) Evaluate $\int \sqrt{4-x^2} \, \mathrm{d}x$
Solution: Let $x = 2\sin\theta$ so that $\mathrm{d}x = 2\cos\theta \, \mathrm{d}\theta$. Then

$$\begin{aligned}\int \sqrt{4-x^2} \, \mathrm{d}x &= \int \sqrt{4 - 4\sin^2\theta} \left(2\cos\theta \, \mathrm{d}\theta\right) \\ &= 2\int \sqrt{4}\sqrt{1-\sin^2\theta} \cos\theta \, \mathrm{d}\theta = 4\int \cos^2\theta \, \mathrm{d}\theta \\ &= 4\int \frac{1+\cos(2\theta)}{2} \mathrm{d}\theta = 2\int \mathrm{d}\theta + \int \cos(2\theta) \, \mathrm{d}(2\theta) \\ &= 2\theta + \sin(2\theta) = 2\theta + 2\sin\theta\cos\theta + C \\ &= \boxed{2\arcsin\left(\frac{x}{2}\right) + x\sqrt{1-\frac{x^2}{4}} + C}\,.\end{aligned}$$

In the last row we used $\sin\theta = \frac{x}{2}$ to get $\theta = \arcsin\left(\frac{x}{2}\right)$, $\cos\theta = \sqrt{1-\sin^2\theta}$.

(2) (Final, 2013) Evaluate $\int_{-1}^{1} \frac{\mathrm{d}x}{(x^2+1)^3}$

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Solution: Recalling $1+\tan^2\theta = \sec^2\theta$ let $x = \tan\theta$ so that $\mathrm{d}x = \sec^2\theta$. We also know hat $\tan\left(\pm\frac{\pi}{4}\right) = \pm 1$ so the integral becomes:

$$\int_{x=-1}^{x=1} \frac{\mathrm{d}x}{\left(x^2+1\right)^3} = \int_{\theta=-\pi/4}^{\theta=\pi/4} \frac{\sec^2\theta}{,\left(\sec^2\theta\right)^3}\,\mathrm{d}\theta = \int_{\theta=-\pi/4}^{\theta=\pi/4} \cos^4\theta\,\mathrm{d}\theta\,.$$

We now use trig integral techniques, here the half-angle formula $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ to get

$$\begin{aligned}
\int_{\theta=-\pi/4}^{\theta=\pi/4} \cos^4\theta\,\mathrm{d}\theta &= \int_{\theta=-\pi/4}^{\theta=\pi/4} \left(\frac{1+\cos(2\theta)}{2}\right)^2 \mathrm{d}\theta \\
&= \int_{\theta=-\pi/4}^{\theta=\pi/4} \left(\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{4}\cos^2(2\theta)\right)\mathrm{d}\theta
\end{aligned}$$

The half-angle formula again gives $\cos^2(2\theta) = \frac{1+\cos(4\theta)}{2}$ so

$$\begin{aligned}
&= \int_{\theta=-\pi/4}^{\theta=\pi/4} \left(\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1+\cos(4\theta)}{8}\right)\mathrm{d}\theta \\
&= \left[\frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta)\right]_{\theta=-\pi/4}^{\theta=\pi/4} \\
&= \frac{3}{16}\pi + \frac{1}{2}\sin(\frac{\pi}{2}) + \frac{1}{32}\sin(\pi) = \boxed{\frac{3}{16}\pi + \frac{1}{2}}\,.
\end{aligned}$$

(3) (105 Final, 2014 + 101 Final, 2009) Convert $\int\left(3-2x-x^2\right)^{-3/2}\mathrm{d}x$ to a trigonometric integral.

Solution: We complete the square: $3-2x-x^2 = 3+1-(1+2x+x^2) = 4-(x+1)^2$. So if we set $x+1 = 2\sin\theta$ we'd have $4-(x+1)^2 = 4-4\sin^2\theta = 4\cos^2\theta$. Since $x = 1+2\sin\theta$ we have $\mathrm{d}x = 2\cos\theta$ and we get

$$\begin{aligned}
\int\left(3-2x-x^2\right)^{-3/2}\mathrm{d}x &= \int\left(4-4\sin^2\theta\right)^{-3/2} 2\cos\theta\,\mathrm{d}\theta \\
&= \frac{2}{4^{3/2}}\int\left(\cos^2\theta\right)^{-3/2}\cos\theta\,\mathrm{d}\theta \\
&= \frac{1}{4}\int\sec^2\theta\,\mathrm{d}\theta
\end{aligned}$$

(4) (Final, 2008) Find the area inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: We need to compute the area between the curves $y = b\sqrt{1-\frac{x^2}{a^2}}$ and $y = -b\sqrt{1-\frac{x^2}{a^2}}$ for $-a \le x \le a$. We therefore substitute $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ to get

$$\begin{aligned}
\text{Area} &= \int_{x=-a}^{x=a} 2b\sqrt{1-\frac{x^2}{a^2}}\,\mathrm{d}x = 2b\int_{\theta=-\pi/2}^{\theta=\pi/2}\sqrt{1-\sin^2\theta}a\cos\theta\,\mathrm{d}\theta \\
&= 2ab\int_{\theta=-\pi/2}^{\theta=\pi/2}\cos^2\theta\,\mathrm{d}\theta \\
&= ab\int_{\theta=-\pi/2}^{\theta=\pi/2}(1+\cos(2\theta))\,\mathrm{d}\theta \\
&= ab\left[\theta + \frac{1}{4}\sin(4\theta)\right]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} = ab\left[\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) + \frac{1}{4}\sin(2\pi) - \frac{1}{4}\sin(-2\pi)\right] \\
&= \boxed{\pi ab}\,.
\end{aligned}$$