Math 101 – SOLUTIONS TO WORKSHEET 13
TRIGONOMETRIC INTEGRALS

Formulas to memorize: $(\sin x)' = \cos x$, $(\cos x)' = \sin x$, $(\tan x)' = \sec^2 x$,

$$\sin(2x) = 2\sin x\cos x \qquad \cos(2x) = 2\cos^2 x - 1 \qquad \cos^2 x = \frac{1+\cos(2x)}{2} \qquad \sin^2 x = \frac{1-\sin(2x)}{x}$$

(1) Evaluate the integrals

(a) $\int \sin^4 x\cos^3 x\,\mathrm{d}x$

Solution: The power of cosine is odd, while the power of sine is even, so let $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$. Then $\sin^4 x = u^4$, $\cos^2 x = 1-u^2$ and

$$\begin{aligned}\int \sin^4 x\cos^3 x\,\mathrm{d}x &= \int u^4(1-u^2)\,\mathrm{d}u = \int \left(u^4-u^6\right)\mathrm{d}u = \frac{u^5}{5}+\frac{u^7}{7}+C\\ &= \frac{1}{5}\sin^5 x+\frac{1}{7}\sin^7 x + C\,.\end{aligned}$$

(b) $\int \sin^5 x\cos^4 x\,\mathrm{d}x$

Solution: This time let $u = \cos x$, $\mathrm{d}u = -\sin x$. Then

$$\begin{aligned}\int \sin^5 x\cos^4 x\,\mathrm{d}x &= \int \left(\sin^2 x\right)^2\cos^4 x\sin x\,\mathrm{d}x\\ &= -\int \left(1-u^2\right)^2 u^4\,\mathrm{d}u = \int \left(1-2u^2+u^4\right)u^4\,\mathrm{d}u\\ &= \int \left(2u^6-u^4-u^8\right)\mathrm{d}u = \frac{2}{7}u^7-\frac{1}{5}u^5-\frac{1}{9}u^9+C\\ &= \frac{2}{7}\cos^7 x-\frac{1}{5}\cos^5 x-\frac{1}{9}\cos^9 x+C\end{aligned}$$

(c) $\int \sin^4 x\cos^4 x\,\mathrm{d}x$

Solution: We have $\sin x\cos x = \frac{1}{2}\sin(2x)$ so

$$\begin{aligned}\int \sin^4 x\cos^4 x\,\mathrm{d}x &= \frac{1}{16}\int \sin^4(2x)\,\mathrm{d}x\\ &= \frac{1}{16}\int \left(\frac{1-\cos(4x)}{2}\right)^2\mathrm{d}x\\ &= \frac{1}{16}\int \left(1-\frac{1}{2}\cos(4x)+\frac{1}{4}\cos^2(4x)\right)\mathrm{d}x\\ &= \frac{1}{16}\left(x-\frac{1}{8}\sin(4x)\right)+\frac{1}{64}\int\frac{1+\cos(8x)}{2}\,\mathrm{d}x\\ &= \frac{1}{16}x-\frac{1}{128}\sin(4x)+\frac{1}{128}x+\frac{1}{64\cdot 8}\sin(8x)+C\,.\end{aligned}$$

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Solution: Use $\sin^2 x = \frac{1-\cos(2x)}{2}$, $\cos^2 x = \frac{1+\cos(2x)}{2}$ to get

$$\begin{aligned}
\int \sin^4 x \cos^4 x \, \mathrm{d}x &= \frac{1}{16}\int (1-\cos(2x))^2 (1+\cos(2x))^2 \\
&= \frac{1}{16}\int ((1-\cos(2x))(1+\cos(2x)))^2 \, \mathrm{d}x \\
&= \frac{1}{16}\int \left(1-\cos^2(2x)\right)^2 \mathrm{d}x \\
&= \frac{1}{16}\int \left(1-2\cos^2(2x)+\cos(4x)\right) \mathrm{d}x \qquad 2\cos^2\theta - 1 = \cos(2\theta) \\
&= \frac{1}{16}\int (\cos(4x)-\cos(8x)) \, \mathrm{d}x \\
&= \frac{1}{64}\sin(4x) - \frac{1}{128}\sin(8x) + C\,.
\end{aligned}$$

(2) Powers of tangent and secant

(a) Evaluate $\int_0^{\pi/4} \tan x \, \mathrm{d}x$

Solution: $\int_0^{\pi/4} \tan x = \int_0^{\pi/4} \frac{\sin x}{\cos x} \mathrm{d}x$. We note that $\sin x \, \mathrm{d}x$ is roughly $d(\cos x) = -\sin x \, \mathrm{d}x$ so letting $u = \cos x$ we have

$$\begin{aligned}
\int_0^{\pi/4} \tan x &= \int_{x=0}^{x=\pi/4} \frac{-\mathrm{d}u}{u} = -\left[\log|u|\right]_{u=1}^{u=\cos(\pi/4)} \\
&= -\left(\log(\frac{1}{\sqrt{2}}) - \log 1\right) = -\log\frac{1}{\sqrt{2}} = \boxed{\frac{1}{2}\log 2}.
\end{aligned}$$

(b) Evaluate $\int_{-\pi/4}^{+\pi/4} \tan x \, \mathrm{d}x$

Solution: Since $\tan(-x) = \tan(x)$ and the domain is symmetric about $x = 0$, the integral vanishes. In detail, let $u = -x$. Then $\mathrm{d}u = -\mathrm{d}x$ abd

$$\begin{aligned}
\int_{x=-\pi/4}^{x=+\pi/4} \tan x \, \mathrm{d}x &= \int_{u=\pi/4}^{u=-\pi/4} \tan(-u)(-\mathrm{d}u) \\
&= \int_{u=\pi/4}^{u=-\pi/4} \tan(u) \, \mathrm{d}u \qquad \tan(-u) = -\tan(u) \\
&= -\int_{u=-\pi/4}^{u=\pi/4} \tan(u) \, \mathrm{d}u \qquad -\int_b^a = \int_a^b \\
&= -\int_{x=-\pi/4}^{x=+\pi/4} \tan(x) \, \mathrm{d}x \qquad \text{substitute } u = x.
\end{aligned}$$