Math 101 – SOLUTIONS TO WORKSHEET 6
SUBSTITUTION

Theorem (Substitution)**.** $\int f'(g(x))g'(x)\,\mathrm{d}x = f(g(x)) + C$. *Equivalently,* $\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$ *where* $u = g(x)$.

(1) Evaluate the integrals

(a) $\int \sin x \cos x\,\mathrm{d}x =$
(hint: use $u = \sin x$)
Solution: Letting $u = \sin x$, $\mathrm{d}u = \sin x\,\mathrm{d}x$ so $\int \sin x \cos x\,\mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\sin^2 x + C}$.

Problem. It's easy to check that $\left(-\frac{1}{4}\cos(2x)\right)' = \frac{1}{2}\sin(2x) = \frac{1}{2}\cdot 2\sin x\cos x = \sin x\cos x$. How is that possible?

(b) (Final, 2014) $\int \cos^3 x \sin^4 x\,\mathrm{d}x =$
Solution: Letting $u = \sin x$, $\mathrm{d}u = \sin x\,\mathrm{d}x$ so

$$\begin{aligned}\int \cos^3 x \sin^4 x\,\mathrm{d}x &= \int \cos^2 x \sin^4 x \cos x\,\mathrm{d}x \\ &= \int (1-u^2)u^4\,\mathrm{d}u \\ &= \frac{u^5}{5} - \frac{u^7}{7} + C \\ &= \boxed{\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C.}\end{aligned}$$

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(c) (Final, 2013) $\int_1^3 (2x-1)e^{x^2-x}\,\mathrm{d}x =$
Solution: Letting $u = x^2 - x$, $\mathrm{d}u = (2x-1)\,\mathrm{d}x$ we get $\int_{x=1}^{x=3}(2x-1)e^{x^2-x}\,\mathrm{d}x = \int_{u=0}^{u=6} e^u\,\mathrm{d}u = [e^u]_{u=0}^{u=6} = e^6 - 1$.
Solution: (Alternative) Using $u = x^2 - x$, $\mathrm{d}u = (2x-1)\,\mathrm{d}x$ we get $\int (2x-1)e^{x^2-x}\,\mathrm{d}x = \int e^u\,\mathrm{d}u = e^u + C = e^{x^2-x} + C$ so

$$\int_{x=1}^{x=3}(2x-1)e^{x^2-x}\,\mathrm{d}x = \left[e^{x^2-x}\right]_{x=1}^{x=3} = \boxed{e^6 - 1}.$$

Solution: (Alternative) Using $u = x^2 - x$, $\mathrm{d}u = (2x-1)\,\mathrm{d}x$ we get

$$\begin{aligned}\int_{x=1}^{x=3}(2x-1)e^{x^2-x}\,\mathrm{d}x &= \int_{x=1}^{x=3} e^u\,\mathrm{d}u = [e^u]_{x=1}^{x=3} \\ &= \left[e^{x^2-x}\right]_{x=1}^{x=3} = e^6 - 1 \\ &= [e^u]_{u=0}^{u=6} = e^6 - 1\,.\end{aligned}$$

Date: 15/1/2016, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(d) (Final, 2012) $\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x =$

Solution: Write this as $\int_0^3 \sqrt{9-x^2}x\,\mathrm{d}x + \int_0^3 \sqrt{9-x^2}\,\mathrm{d}x$. The second term is the area of a quarter-circle of radius 3, so is $\frac{9}{4}\pi$. For the first term we use $u = 9 - x^2$, $\mathrm{d}u = -2x\,\mathrm{d}x$ to see that

$$\begin{aligned}\int_{x=0}^{x=3} \sqrt{9-x^2}x\,\mathrm{d}x &= \int_{u=9}^{u=0} \sqrt{u}\left(-\frac{1}{2}\right)\mathrm{d}u = \frac{1}{2}\int_{u=0}^{u=9} u^{1/2}\,\mathrm{d}u \\ &= \frac{1}{2}\frac{2}{3}\left[u^{3/2}\right]_{u=0}^{u=9} = \frac{1}{3}9^{3/2} = 9\,.\end{aligned}$$

In conclusion, $\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x = \boxed{9+\frac{9}{4}\pi}$.