MATH 101: COMPUTING ANTI-DERIVATIVES BY MASSAGING

LIOR SILBERMAN

In this note I collect a few examples of computing indefinite integrals by "massaging" a function whose derivative is similar to the desired result.

. The writing is pedagogical (illustrating thinking) rather than exam-motivated.

Problem 1. Compute $\int 7e^{-x/3}\,\mathrm{d}x$.

Solution: We recall that $(e^x)' = e^x$, so we try $e^{-x/3}$, where we get $\left(e^{-x/3}\right)' = -\frac{1}{3}e^{-x/3}$. Solving for $e^{-x/3}$ we find

$$e^{-x/3} = -3\left(e^{-x/3}\right)' = \left(-3e^{-x/3}\right)'$$

so

$$\int e^{-x/3}\,\mathrm{d}x = -3e^{-x/3} + C\,.$$

Problem 2. Compute $\int \frac{1}{1+4x^2}\,\mathrm{d}x$.

Solution: We recall that $(\arctan x)' = \frac{1}{1+x^2}$. Since we need to get $\frac{1}{1+(2x)^2}$ we try $\arctan(2x)$. By the chain rule, $(\arctan(2x))' = 2\cdot\frac{1}{1+(2x)^2} = \frac{2}{1+4x^2}$ so solving for $\frac{1}{1+4x^2}$ we see:

$$\int \frac{1}{1+4x^2}\,\mathrm{d}x = \frac{1}{2}\arctan(2x) + C\,.$$

Problem 3. Compute $\int \frac{\mathrm{d}x}{3x+1}$.

Solution 1: We remember that $(\log|x|)' = \frac{1}{x}$ so we try $\log|3x+1|$. By the chain rule, $(\log|3x+1|)' = \frac{3}{3x+1}$, so we divide by 3 to get

$$\frac{1}{3x+1} = \frac{1}{3}(\log|3x+1|)' = \left(\frac{1}{3}\log|3x+1|\right)'$$

so

$$\int \frac{\mathrm{d}x}{3x+1} = \frac{1}{3}\log|3x+1| + C\,.$$

Solution 2: We remember note that $\frac{1}{3x+1} = \frac{1}{3}\cdot\frac{1}{x+1/3}$. Again $(\log|x|)' = \frac{1}{x}$ but this also means $\left(\log\left|x+\frac{1}{3}\right|\right)' = \frac{1}{x+1/3}$ and we get:

$$\frac{1}{3x+1} = \frac{1}{3}\left(\log\left|x+\frac{1}{3}\right|\right)' = \left(\frac{1}{3}\log\left|x+\frac{1}{3}\right|\right)'$$

so

$$\int \frac{\mathrm{d}x}{3x+1} = \frac{1}{3}\log\left|x+\frac{1}{3}\right| + C\,.$$

Problem. $\frac{1}{3}\log|3x+1|$ and $\frac{1}{3}\log\left|x+\frac{1}{3}\right|$ are different. How is that possible?

Answer: Both are correct. $\frac{1}{3}\log|3x+1| = \frac{1}{3}\log\left|3\cdot\left(x+\frac{1}{3}\right)\right| = \frac{1}{3}\left(\log 3 + \log\left|x+\frac{1}{3}\right|\right)$ so the two anti-derivatives differ by the constant $\frac{1}{3}\log 3$ – which is how things should be!

Problem 4. Compute $\int \sin x \cos x \, \mathrm{d}x$.

Solution: By the half-angle formula $2 \sin x \cos x = \sin(2x)$ this is $\int \frac{1}{2} \sin(2x) \, \mathrm{d}x$. We know that $(\cos x)' = -\sin x$; replacing x with $2x$ gives $(\cos(2x))' = -2\sin(2x)$ and solving for $\sin(2x)$ we get

$$\frac{1}{2}\sin(2x) = -\frac{1}{4}(\cos(2x))' = \left(-\frac{1}{4}\cos(2x)\right)'$$

so

$$\int \sin x \cos x \, \mathrm{d}x = -\frac{1}{4}\cos(2x) + C\,.$$