MATH 100 – WORKSHEET 11
EXPONENTIAL GROWTH AND DECAY

1. Exponentials

Growth/decay described by the *differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}y} = ky\,,$$

Solution: $y =$

(1) A pair of invasive Opossums arrives in BC in 1930. Unchecked, Opossums can triple their population annually. Assume that this in fact happens.
   (a) At what time will there be 1000 Opossums in BC? 10,000 Opossums?
   Solution: $\frac{\log 500}{\log 3}$, $\frac{\log 5000}{\log 3}$.
   (b) Write a differential equation expressing the growth of the Opossum population with time.
   Solution: $\frac{\mathrm{d}y}{\mathrm{d}t} = (\log 3)y$.

(2) A radioactive sample decays according to the law

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -km\,.$$

   (a) Suppose that one-quarter of the sample remains after 10 hours. What is the half-life?
   Solution: 5 hours.
   (b) A 100-gram sample is left unattended for three days. How much of it remains?
   Solution: $100 \cdot e^{-\frac{\log 2}{5} \cdot 3 \cdot 24} = 100 \cdot 2^{-\frac{72}{5}} \approx 4.63\text{g}$

(3) Euler found that the tension in a wire wound around a cylinder inreases according to the equation

$$\frac{\mathrm{d}T}{\mathrm{d}\alpha} = \mu T$$

   where μ is the coefficient of friction and α is the angle around the cylinder.
   (a) When mooring a large ship a cable is wound around a bollard. It is found that when looping the cable once around the bollard, the ratio of tensions at the two ends of the rope is 20. What is the coefficient of friction?
   (b) The rope is wound 3.5 times around the bollard. What is the force gain?

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2. Newton's Law of Cooling

Fact 1. *When a body of temperature T is placed in an environment of temperature T_{env}, the rate of change of T is* negatively proportional *to the temperature difference $T - T_0$. In other words, there is k such that*

$$T' = -k(T - T_{env})\,.$$

- *key idea*: change variables to the *temperature difference*. Let $y = T - T_{\text{env}}$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} - 0 = -ky$$

so there is C for which

$$y(t) = Ce^{-kt}\,.$$

Solving for T we get:

$$T(t) = T_{\text{env}} + Ce^{-kt}\,.$$

Setting $t = 0$ we find $T(0) = T_{\text{env}} + C$ so $C = T(0) - T_{\text{env}}$ and

$$T(t) = T_{\text{env}} + (T(0) - T_{\text{env}})e^{-kt}\,.$$

Corollary 2. $\lim_{t\to\infty} y(t) = T_0$.

Example (Final Exam, 2010)**.** When an apple is taken from a refrigerator, its temperature is $3°C$. After 30 minutes in a $19°C$ room its temperature is $11°C$.

(1) Write the *differential equation* satisfied by the temperature $T(t)$ of the apple.
(2) Find the temperature of the apple 90 minutes after it is taken from the refrigerator, expressed as an integer number of degrees Celsius.
(3) Determine the time when the temperature of the apple is $16°C$.

Solution:

(1) Let $y(t) = T(t) - T_{\text{env}}$ be the temperature difference. Then $y(0) = -16°C$, $y(30) = -8°C$. We know that $y(t) = y(0)e^{-kt}$ for some k, so that $-8 = (-16)e^{-30k}$ so

$$e^{-30k} = \frac{-8}{-16} = \frac{1}{2}\,.$$

Taking logarithms we find

$$-30k = \log\frac{1}{2} = -\log 2$$

so

$$k = \frac{\log 2}{30}\,.$$

We thus get $\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} = -ky = -\frac{\log 2}{30}(T - 19°C)$.

(2) We have $y(90) = -16\cdot e^{-\frac{\log 2}{30}\cdot 90} = -16\cdot e^{-3\log 2} = -16\cdot\left(e^{\log 2}\right)^{-3} = -16\cdot\frac{1}{2^3} = -\frac{16}{8} = -2$ so $T(90) = T_{\text{env}} + (-2) = 17°C$.

(3) If $T(t) = 16°C$ then $y(t) = -3°C$, so we need t such that

$$-3 = -16\cdot e^{-\frac{\log 2}{30}\cdot t}$$

or

$$\frac{3}{16} = e^{-\frac{\log 2}{30}t}\,.$$

Taking logarithms we get

$$\log 3 - \log 16 = \log\frac{3}{16} = -\frac{\log 2}{30}t$$

so

$$t = \frac{\log 16 - \log 3}{\log 2}30\text{min}$$