MATH 100: MORE EXAMPLES OF INEQUALITIES

(1) Show that $\sqrt{1+x} < 1 + \frac{x}{2}$ for all $x > 0$.

Solution 1: (scratchwork) $\sqrt{1+x} < 1+\frac{x}{2} \iff \sqrt{1+x}-1 < +\frac{x}{2} \iff \frac{\sqrt{1+x}-1}{x} < \frac{1}{2}$ AHA!

(solution) Let $f(u) = \sqrt{u}$. Then we need to compare $f(1+x)$ and $f(1)$. By the MVT we there os

$$\frac{\sqrt{1+x}-1}{x} = \frac{f(1+x)-f(1)}{1+x-1} = f'(c)$$

for some $c \in (1, 1+x)$ (since $x > 0$, $1+x > 1$). But $f'(c) = \frac{1}{2\sqrt{c}} < \frac{1}{2}$ since $c > 1$, so

$$\frac{\sqrt{1+x}-1}{x} < \frac{1}{2}\,.$$

Multiplying by $x > 0$ we get

$$\sqrt{1+x}-1 < \frac{x}{2}$$

and adding 1 we finally see

$$\sqrt{1+x} < 1 + \frac{x}{2}\,.$$

Solution 2: Let $g(x) = \sqrt{1+x}$. Then $g'(x) = \frac{1}{2\sqrt{1+x}}$ so $g(0) = 1$, $g'(0) = \frac{1}{2}$ and $T_1(x) = 1 + \frac{x}{2}$ is the linear approximation to $g(x)$ near $a = 0$. By the Lagrange Remainder Formula there is c in $(0, x)$ such that

$$g(x) - T_1(x) = R_1(x) = \frac{g''(c)}{2!}x^2 = -\frac{1}{4(1+c)^{3/2}}x^2 < 0\,.$$

it follows that

$$\sqrt{1+x} = g(x) < T_1(x) = 1 + \frac{x}{2}\,.$$

Solution 3: Let $h(x) = 1+\frac{x}{2}-\sqrt{1+x}$. We have $h(0) = 1+\frac{0}{2}-\sqrt{1+0} = 0$. Also, h is differentiable on $(-1,\infty)$ with and

$$h'(x) = \frac{1}{2} - \frac{1}{2\sqrt{1+x}} = \frac{1}{2}\left[1 - \frac{1}{\sqrt{1+x}}\right]\,.$$

Now for $x > 0$, $\sqrt{1+x} > 1$ so $\frac{1}{\sqrt{1+x}} < 1$ and $h'(x) > 0$. It follows that h is strictly increasing on $(0, \infty)$ so that $h(x) > 0$ if $x > 0$. But this means

$$1 + \frac{x}{2} - \sqrt{1+x} > 0$$

so

$$1 + \frac{x}{2} > \sqrt{1+x}\,.$$