Math 412: Problem Set 1 (due 15/1/2014)

Practice problems, any sub-parts marked "OPT" (optional) and supplementary problems are not for submission.

Practice problems

P1 Show that the map $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = x - 2y + z$ is a linear map. Show that the maps $(x,y,z) \mapsto 1$ and $(x,y,z) \mapsto x^2$ are non-linear.

P2 Let F be a field, X a set. Carefully show that pointwise addition and scalar multiplication endow the set F^X of functions from X to F with the structure of an F-vectorspace.

For submission

RMK The following idea will be used repeatedly during the course to prove that sets of vectors are linearly independent. Make sure you understand how this argument works.

1. Let V be a vector space, $S \subset V$ a set of vectors. A *minimal dependence* in S is an equality $\sum_{i=1}^{m} a_i \underline{v}_i = \underline{0}$ where $\underline{v}_i \in S$ are distinct, a_i are scalars not all of which are zero, and $m \geq 1$ is as small as possible so that such $\{a_i\}$, $\{\underline{v}_i\}$ exist.
   (a) Find a minimal dependence among $\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix} \right\} \subset \mathbb{R}^3$.
   (b) Show that in a minimal dependence the a_i are all non-zero.
   (c) Suppose that $\sum_{i=1}^{m} a_i \underline{v}_i$ and $\sum_{i=1}^{m} b_i \underline{v}_i$ are minimal dependences in S, involving the exact same set of vectors. Show that there is a non-zero scalar c such that $a_i = cb_i$.
   (d) Let $T\colon V \to V$ be a linear map, and let $S \subset V$ be a set of (non-zero) eigenvectors of T, each corresponding to a distinct eigenvalue. Applying T to a minimal dependence in S obtain a contradiction to (b) and conclude that S is actually linearly independent.
   (**e) Let Γ be a group. The set $\mathrm{Hom}(\Gamma, \mathbb{C}^\times)$ of group homomorphisms from Γ to the multiplicative group of nonzero complex numbers is called the set of *quasicharacters* of Γ (the notion of "character of a group" has an additional, different but related meaning, which is not at issue in this problem). Show that $\mathrm{Hom}(\Gamma, \mathbb{C}^\times)$ is linearly independent in the space $\mathbb{C}^\Gamma$ of functions from Γ to $\mathbb{C}$.

2. Let $S = \{\cos(nx)\}_{n=0}^{\infty} \cup \{\sin(nx)\}_{n=1}^{\infty}$, thought of as a subset of the space $C(-\pi,\pi)$ of continuous functions on the interval $[-\pi,\pi]$.
   (a) Applying $\frac{d}{dx}$ to a putative minimal dependence in S obtain a different linear dependence of at most the same length, and use that to show that S is, in fact, linearly independent.
   (b) Show that the elements of S are an orthogonal system with respect to the inner product $\langle f,g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x$ (feel free to look up any trig identities you need). This gives a different proof of their independence.
   (c) Let $W = \mathrm{Span}_{\mathbb{C}}(S)$ (this is usually called "the space of trigonometric polynomials"; a typical element is $5 - \sin(3x) + \sqrt{2}\cos(15x) - \pi\cos(32x)$). Find a ordering of S so that the matrix of the linear map $\frac{d}{dx}\colon W \to W$ in that basis has a simple form.

3. (Matrices associated to linear maps) Let V, W be vector spaces of dimensions n, m respectively. Let $T \in \operatorname{Hom}(V,W)$ be a linear map from V to W. Show that there are ordered bases $B = \{\underline{v}_j\}_{j=1}^n \subset V$ and $C = \{\underline{w}_i\}_{i=1}^m \subset W$ and an integer $d \leq \min\{n,m\}$ such that the matrix $A = (a_{ij})$ of T with respect to those bases satisfies $a_{ij} = \begin{cases} 1 & i = j \leq d \\ 0 & \text{otherwise} \end{cases}$, that is has the form

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

(Hint1: study some examples, such as the matrices $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ and $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$) (Hint2: start your solution by choosing a basis for the image of T).

Extra credit: Finite fields

4. Let F be a field.
   (a) Define a map $\iota\colon (\mathbb{Z},+) \to (F,+)$ by mapping $n \in \mathbb{Z}_{\geq 0}$ to the sum $1_F + \cdots + 1_F$ n times. Show that this is a group homomorphism.
   DEF If the map ι is injective we say that F is of *characteristic zero*.
   (b) Suppose there is a non-zero $n \in \mathbb{Z}$ in the kernel of ι. Show that the smallest positive such number is a prime number p.
   DEF In that case we say that F is of *characteristic p*.
   (c) Show that in that case ι induces an isomorphism between the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and a subfield of F. In particular, there is a unique field of p elements up to isomorphism.

5. Let F be a field with finitely many elements.
   (a) Carefully endow F with the structure of a vector space over $\mathbb{F}_p$ for an appropriately chosen p.
   (b) Show that there exists an integer $r \geq 1$ such that F has p^r elements.
   RMK For every prime power $q = p^r$ there is a field $\mathbb{F}_q$ with q elements, and two such fields are isomorphic. They are usually called *finite fields*, but also *Galois fields* after their discoverer.

Supplementary Problems I: A new field

A. Let $\mathbb{Q}(\sqrt{2})$ denote the set $\left\{a + b\sqrt{2} \mid a,b \in \mathbb{Q}\right\} \subset \mathbb{R}$.
   (a) Show that $\mathbb{Q}(\sqrt{2})$ is a $\mathbb{Q}$-subspace of $\mathbb{R}$.
   (b) Show that $\mathbb{Q}(\sqrt{2})$ is two-dimensional as a $\mathbb{Q}$-vector space. In fact, identify a basis.
   (*c) Show that $\mathbb{Q}(\sqrt{2})$ is a field.
   (**d) Let V be a vector space over $\mathbb{Q}(\sqrt{2})$ and suppose that $\dim_{\mathbb{Q}(\sqrt{2})} V = d$. Show that $\dim_{\mathbb{Q}} V = 2d$.

Supplementary Problems II: How physicists define vectors

Fix a field F.

B. (The general linear group)
   (a) Let $\mathrm{GL}_n(F)$ denote the set of invertible $n \times n$ matrices with coefficients in F. Show that $\mathrm{GL}_n(F)$ forms a group with the operation of matrix multiplication.
   (b) For a vector space V over F let $\mathrm{GL}(V)$ denote the set of invertible linear maps from V to itself. Show that $\mathrm{GL}(V)$ forms a group with the operation of composition.
   (c) Suppose that $\dim_F V = n$ Show that $\mathrm{GL}_n(F) \simeq \mathrm{GL}(V)$ (hint: show that each of the two group is isomorphic to $\mathrm{GL}(F^n)$).

C. (Group actions) Let G be a group, X a set. An *action* of G on X is a map $\cdot \colon G \times X \to X$ such that $g \cdot (h \cdot x) = (gh) \cdot x$ and $1_G \cdot x = x$ for all $g, h \in G$ and $x \in X$ (1_G is the identity element of G).
   (a) Show that matrix-vector multiplication $(g, \underline{v}) \mapsto g\underline{v}$ defines an action of $G = \mathrm{GL}_n(F)$ on $X = F^n$.
   (b) Let V be an n-dimensional vector space over F, and let $\mathcal{B}$ be the set of ordered bases of V. For $g \in \mathrm{GL}_n(F)$ and $B = \{\underline{v}_i\}_{i=1}^{\dim V} \in \mathcal{B}$ set $gB = \left\{\sum_{j=1}^n g_{ij}\underline{v}_i\right\}_{j=1}^n$. Check that $gB \in \mathcal{B}$ and that $(g, B) \mapsto gB$ is an action of $\mathrm{GL}_n(F)$ on $\mathcal{B}$.
   (c) Show that the action is *transitive*: for any $B, B' \in \mathcal{B}$ there is $g \in \mathrm{GL}_n(F)$ such that $gB = B'$.
   (d) Show that the action is *simply transitive*: that the g from part (b) is unique.

D. (From the physics department) Let V be an n-dimensional vector space, and let $\mathcal{B}$ be its set of bases. Given $\underline{u} \in V$ define a map $\phi_{\underline{u}} \colon \mathcal{B} \to F^n$ by setting $\phi_{\underline{u}}(B) = \underline{a}$ if $B = \{\underline{v}_i\}_{i=1}^n$ and $\underline{u} = \sum_{i=1}^n a_i \underline{v}_i$.
   (a) Show that $\alpha\phi_{\underline{u}} + \phi_{\underline{u}'} = \phi_{\alpha\underline{u}+\underline{u}'}$. Conclude that the set $\left\{\phi_{\underline{u}}\right\}_{\underline{u}\in V}$ forms a vector space over F.
   (b) Show that the map $\phi_{\underline{u}} \colon \mathcal{B} \to F^n$ is *equivariant* for the actions of B(a),B(b), in that for each $g \in \mathrm{GL}_n(F)$, $B \in \mathcal{B}$, $g\left(\phi_{\underline{u}}(B)\right) = \phi_{\underline{u}}(gB)$.
   (c) Physicists define a "covariant vector" to be an equivariant map $\phi \colon \mathcal{B} \to F^n$. Let Φ be the set of covariant vectors. Show that the map $\underline{u} \mapsto \phi_{\underline{u}}$ defines an isomorphism $V \to \Phi$. (Hint: define a map $\Phi \to V$ by fixing a basis $B = \{\underline{v}_i\}_{i=1}^n$ and mapping $\phi \mapsto \sum_{i=1}^n a_i \underline{v}_i$ if $\phi(B) = \underline{a}$).
   (d) Physicists define a "contravariant vector" to be a map $\phi \colon \mathcal{B} \to F^n$ such that $\phi(gB) = {}^t g^{-1} \cdot (\phi(B))$. Verify that $(g, \underline{a}) \mapsto {}^t g^{-1}\underline{a}$ defines an action of $\mathrm{GL}_n(F)$ on F^n, that the set Φ' of contravariant vectors is a vector space, and that it is naturally isomorphic to the dual vector space V' of V.

Supplementary Problems III: Fun in positive characteristic

E. Let F be a field of characterstic 2 (that is, $1_F + 1_F = 0_F$).
   (a) Show that for all $x, y \in F$ we have $x + x = 0_F$ and $(x+y)^2 = x^2 + y^2$.
   (b) Considering F as a vector space over $\mathbb{F}_2$ as in 6(a), show that the map given by $\mathrm{Frob}(x) = x^2$ is a linear map.
   (c) Suppose that the map $x \mapsto x^2$is actually F-linear and not only $\mathbb{F}_2$-linear. Show that $F = \mathbb{F}_2$.
   RMK Compare your answer with practice problem 1.

F. (This problem requires a bit of number theory) Now let F have characteristic $p > 0$. Show that the *Frobenius endomorphism* $x \mapsto x^p$ is $\mathbb{F}_p$-linear.