MATH 253 – WORKSHEET 20
ITERATED INTEGRALS ON RECTANGLES

Theorem (Fubini)**.** *Let $f(x,y)$ be integrable on the rectangle $R=[a,b]\times[c,d]$. Then*

$$\boxed{\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_{y=c}^{y=d}\mathrm{d}y\left(\int_{x=a}^{x=b}\mathrm{d}x\,f(x,y)\right)=\int_{x=a}^{x=b}\mathrm{d}x\left(\int_{y=c}^{y=d}\mathrm{d}y\,f(x,y)\right)}$$

(1) Integrate $f(x,y)=(1-y)x$ on $[2,3]\times[4,5]$.
Solution: $\iint\limits_{[2,3]\times[4,5]}(1-y)x\,\mathrm{d}x\,\mathrm{d}y=\int_{y=4}^{y=5}\mathrm{d}y\int_{x=2}^{x=3}\mathrm{d}x(1-y)x=\int_{y=4}^{y=5}\mathrm{d}y(1-y)\int_{x=2}^{x=3}\mathrm{d}x x$ since $(1-y)$ is constant in the inner integral. Therefore:

$$\begin{aligned}\iint\limits_{[2,3]\times[4,5]}(1-y)x\,\mathrm{d}x\,\mathrm{d}y &= \left(\int_{y=4}^{y=5}\mathrm{d}y(1-y)\right)\cdot\left(\int_{x=2}^{x=3}\mathrm{d}x x\right)\\ &= \left[-\frac{1}{2}(1-y)^2\right]_{y=4}^{y=5}\cdot\left[\frac{1}{2}x^2\right]_{x=2}^{x=3}\\ &= -\frac{1}{4}\,[16-9]\,[9-4]=-\frac{35}{4}\,.\end{aligned}$$

Note the general feature that

$$\boxed{\iint\limits_{[a,b]\times[c,d]}f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y=\left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_a^b g(y)\,\mathrm{d}y\right)\,.}$$

(2) Integrate $f(x,y)=x(y+x^2)$ on $R=[0,1]\times[0,1]$.
Solution 1: By Fubini, $\iint_R x\left(y+x^2\right)\mathrm{d}x\,\mathrm{d}y=$

$$\begin{aligned}&= \int_{x=0}^{x=1}\mathrm{d}x x\int_{y=0}^{y=1}(y+x^2)\,\mathrm{d}y\\ &= \int_{x=0}^{x=1}\mathrm{d}x x\left[\frac{1}{2}y^2+yx^2\right]_{y=0}^{y=1}\\ &= \int_{x=0}^{x=1}\mathrm{d}x x\left(x^2+\frac{1}{2}\right)\\ &= \left[\frac{x^2}{4}+\frac{x^4}{4}\right]_{x=0}^{x=1}=\frac{1}{2}\,.\end{aligned}$$

Solution 2: $\iint_R x\left(y+x^2\right)\mathrm{d}x\,\mathrm{d}y=\iint_R xy\,\mathrm{d}x\,\mathrm{d}y+\iint_R x^3\,\mathrm{d}x\,\mathrm{d}y$. Now apply Fubini to see the integral is

$$\left(\int_0^1 x\,\mathrm{d}x\right)\left(\int_0^1 y\,\mathrm{d}y\right)+\left(\int_0^1 x^3\,\mathrm{d}x\right)\left(\int_0^1 1\,\mathrm{d}y\right)=\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{4}\cdot 1=\frac{1}{2}\,.$$

(3) Evaluate $\iint_{[-1,1]\times[0,1]}\frac{y\sin x}{1+\cos^2 y}\,\mathrm{d}x\,\mathrm{d}y$. What is the integral of $f(x,y)=(x+y)\,e^{-x^4-y^4}$ on the plane?
Solution: The first function is odd in x, and the domain is symmetric in x, so the integral is zero. The second function is odd under rotating the plane by π (changing $x\to -x$, $y\to -y$) so

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again the integral is zero. For the second function can also write it as $xe^{-x^4-y^4} + ye^{-x^4-y^4}$ where the first term is odd in x, the second in y so each integrates to zero separately.

(4) Find the average value of $f(x,y) = e^y\sqrt{x+e^y}$ over the rectangle $[0,4]\times[0,1]$.

Solution: The average is the integral over the domain divided by the area of the domain, that is

$$\begin{aligned}
\frac{1}{4}\int_{x=0}^{x=4} \mathrm{d}x \int_{y=0}^{y=1} \sqrt{x+e^y}e^y\,\mathrm{d}y &\overset{u=e^y}{=} \frac{1}{4}\int_{x=0}^{x=4}\mathrm{d}x\int_{u=1}^{u=e}\sqrt{x+u}\,\mathrm{d}u \\
&= \frac{1}{4}\left[\frac{2}{3}(x+u)^{3/2}\right]_{u=1}^{u=e} \\
&= \frac{1}{6}\int_{x=0}^{x=4}\mathrm{d}x\left[(x+e)^{3/2}-(x+1)^{3/2}\right] \\
&= \frac{1}{6}\left[\frac{2}{5}(x+e)^{5/2}-\frac{2}{5}(x+1)^{5/2}\right]_{x=0}^{x=4} \\
&= \frac{1}{15}\left[(4+e)^{5/2}-5^{5/2}-e^{5/2}+1\right].
\end{aligned}$$