Solutions



**Q1** [9 marks]

Find the partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, and $\frac{\partial^2 z}{\partial x \partial y}$ where $z$ is defined (possibly implicitly) as a function of $x$ and $y$ by the equations below.

(a) $z = x\cos(xy)$

$$\frac{\partial z}{\partial x} = \cos(xy) - xy\sin(xy)$$

$$\frac{\partial z}{\partial y} = -x^2\sin(xy)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -2x\sin(xy) - x^2 y\cos(xy)$$

(b) $z = x^2 \int_0^y \sin^8(\sqrt{t})dt$

$$\frac{\partial z}{\partial x} = 2x\int_0^y \sin^8(\sqrt{t})dt$$

$$\frac{\partial z}{\partial y} = x^2\sin^8(\sqrt{y})$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x\sin^8(\sqrt{y})$$

(c) $z + e^z = x^2 + y^2$

$$\frac{\partial}{\partial x}\left(z + e^z = x^2 + y^2\right)$$

$$\frac{\partial z}{\partial x} + e^z \frac{\partial z}{\partial x} = 2x + 0$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{2x}{1 + e^z}}$$

$$\frac{\partial}{\partial y}\left(z + e^z = x^2 + y^2\right) \Rightarrow \frac{\partial z}{\partial y} + e^z \frac{\partial z}{\partial y} = 2y$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{2y}{1 + e^z}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{2y}{1 + e^z}\right)$$

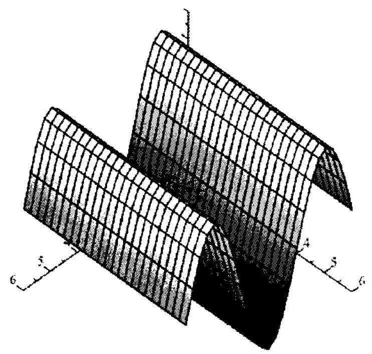
$$= \frac{-2y}{(1 + e^z)^2} e^z \frac{\partial z}{\partial x}$$

$$= \frac{-2y}{(1 + e^z)^2} e^z \frac{2x}{1 + e^z}$$

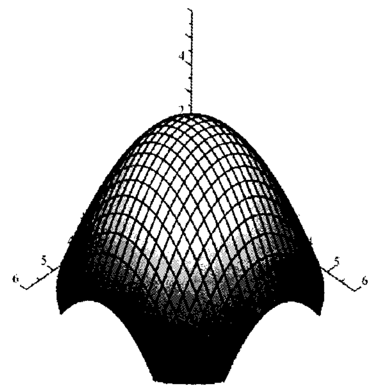
$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{-4xy e^z}{(1 + e^z)^3}}$$

**Q2** [9 marks]

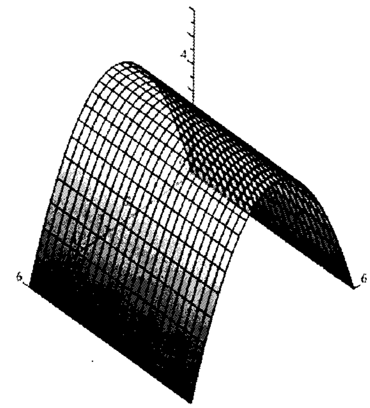
Put the letter of each graph in the box below the corresponding contour plot on the next page. The axes of the below graphs are all oriented in the standard way: the positive $x$-axis is on the left, the positive $y$-axis is on the right, and the positive $z$-axis is upward.



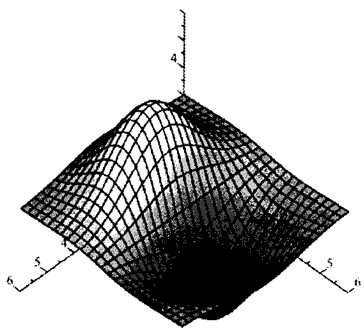
A



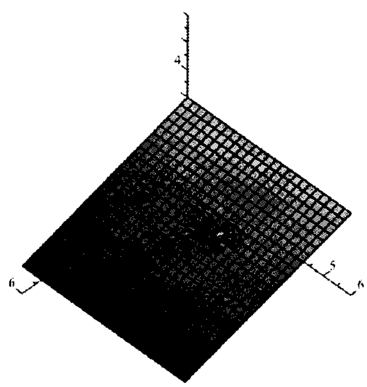
B



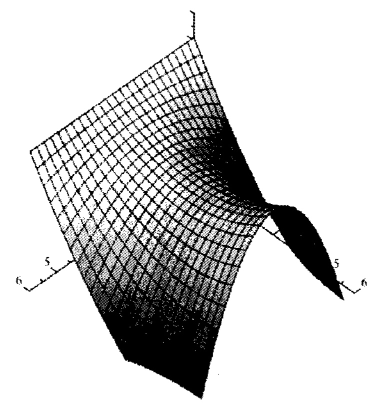
C



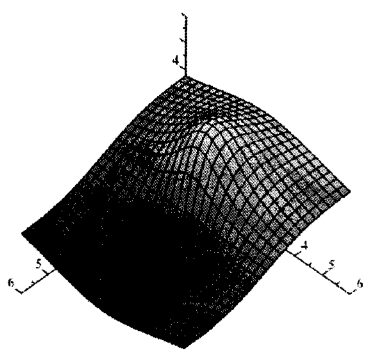
D



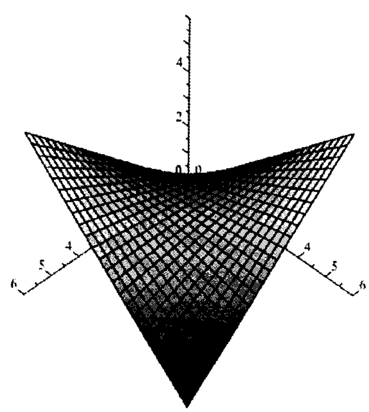
E



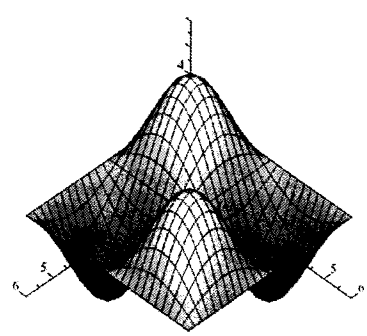
F



G

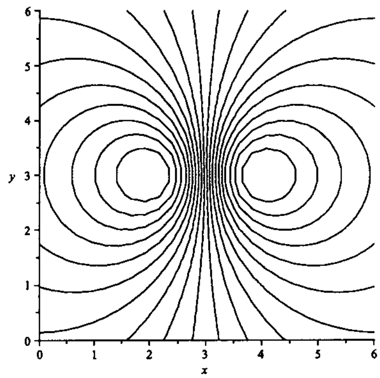


H

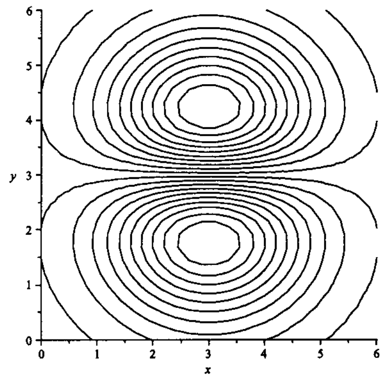


I

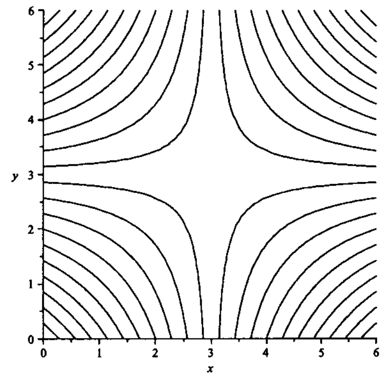
In the below contour plots, the *values* of the contours are evenly spaced. Put the letter of the corresponding graph in the box below the contour plot.



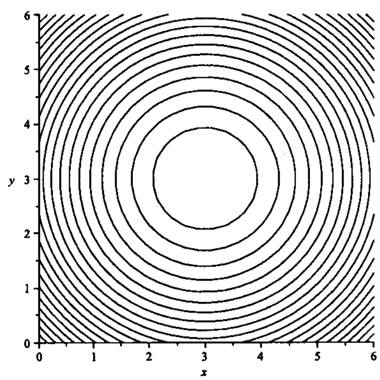
G



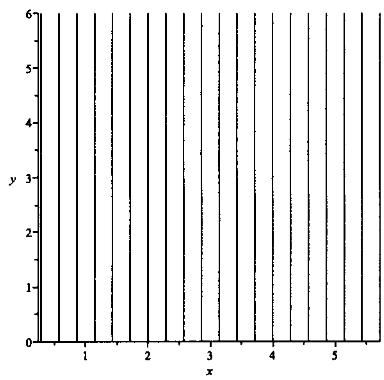
D



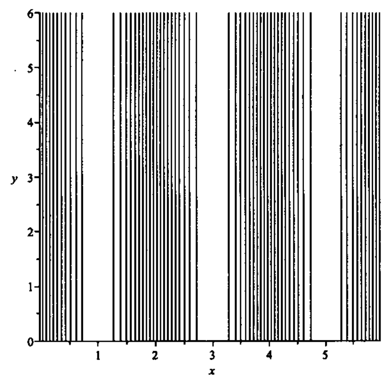
H



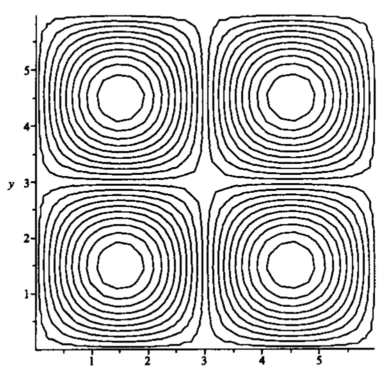
B



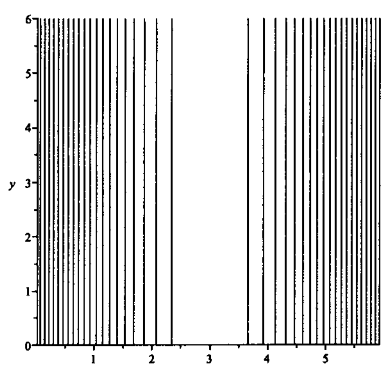
E



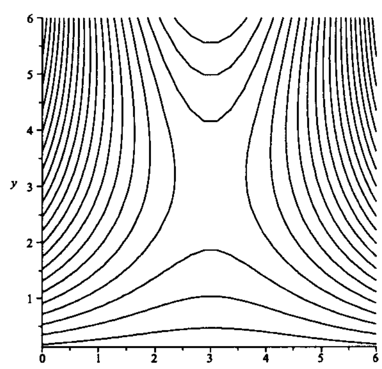
A



I



C



F

**Q3** [8 marks]

Let $f(x, y) = \sqrt{1 + x^2y^3}$.

(*a*) Find an equation of the plane tangent to the surface $z = f(x, y)$ at the point $(1, 2, 3)$.

$z = 3 + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

$f_x = \dfrac{xy^3}{\sqrt{1+x^2y^3}}$

$\boxed{z = 3 + \frac{8}{3}(x-1) + 2(y-2)}$

$f_y = \dfrac{3x^2y^2}{2\sqrt{1+x^2y^3}}$

or

$f_x(1,2) = \frac{8}{3}$

$z = \frac{8}{3}x + 2y - \frac{11}{3}$ or $11 = 8x + 6y - 3z$

$f_y(1,2) = \dfrac{3\cdot 4}{2\cdot 3} = 2$

(*b*) Use your answer in part (a) to find an approximate value for $f(1.03, 1.98)$.

$$f(1.03, 1.98) \approx 3 + \frac{8}{3}(1.03-1) + 2(1.98-2)$$

$$= 3 + \frac{8}{3}(0.03) + 2(-0.02)$$

$$= 3 + 0.08 - 0.04$$

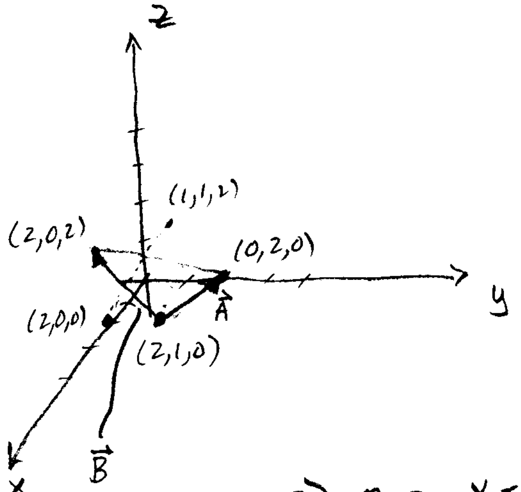
$$= 3.04$$

$$\boxed{f(1.03, 1.98) \approx 3.04}$$

**Q4** [9 marks]

Let $P$ be the plane passing through the points $(2,1,0)$, $(2,0,2)$, and $(0,2,0)$ and let $L$ be the line passing through the points $(2,0,0)$ and $(1,1,2)$.

($a$) Find the equation of the plane $P$. Simplify your answer and write it in the form $z = f(x,y)$.




$\vec{A} = \langle -2, 1, 0 \rangle$

$\vec{B} = \langle 0, -1, 2 \rangle$

Normal vector to plane: $\vec{N} = \vec{A} \times \vec{B}$

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 0 \\ 0 & -1 & 2 \end{vmatrix} = \langle 2, 4, 2 \rangle$$

$$0 = 2(x-2) + 4(y-1) + 2z$$

$$\Rightarrow 0 = x - 2 + 2(y-1) + z = x + 2y + z - 4 \Rightarrow \boxed{z = -x - 2y + 4}$$

($b$) Find the point of intersection of the plane $P$ and the line $L$.

line passes through $(2,0,0)$ and has direction $\langle -1, 1, 2 \rangle$

$\Rightarrow$ $x = 2 - t$, $y = t$, $z = 2t$ is a parameterization of the line.

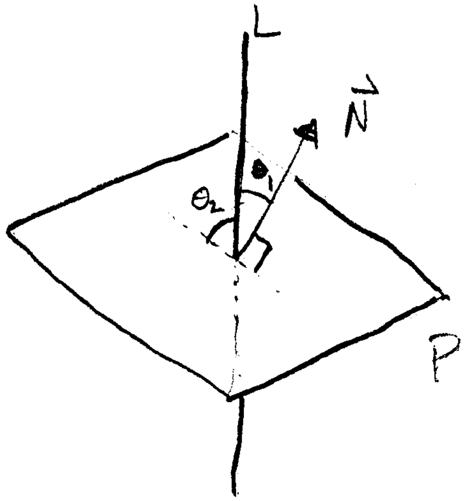
the point of intersection with $z = -x - 2y + 4$ occurs when $t$ satisfies $2t = -2 + t - 2t + 4$

$\Rightarrow 3t = 2$ $\quad t = \frac{2}{3}$ $\quad$ so $x = \frac{4}{3}$ $\quad y = \frac{2}{3}$ $\quad z = \frac{4}{3}$

$$\boxed{\left(\frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right)}$$

(c) Find the angle between the plane $P$ and the line $L$.

The angle between the line and the normal to the plane is given by




$$\theta_1 = \cos^{-1}\left(\frac{\langle -1,1,2\rangle \cdot \langle 2,4,2\rangle}{|\langle -1,1,2\rangle|\,|\langle 2,4,2\rangle|}\right)$$

$$= \cos^{-1}\left(\frac{-2+4+4}{\sqrt{6}\cdot\sqrt{24}}\right)$$

$$= \cos^{-1}\left(\frac{6}{\sqrt{6}\cdot 2\sqrt{6}}\right) = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta_1 = \frac{\pi}{3}$$

The angle between the line and the plane is the complimentary angle $\theta_2 = \frac{\pi}{2} - \theta_1$ (see picture).

so

$$\theta_2 = \boxed{\frac{\pi}{6}}$$

# SOLUTION TO PROBLEM 4

(1) Let $P$ be the plane passing through the points $(2,1,0)$, $(2,0,2)$ and $(0,2,0)$. Let $L$ be the line passing through the points $(2,0,0)$ and $(1,1,2)$.

(a) Find an equation for the plane $P$. Simplify your answer and write it in the form $z = f(x,y)$.

**Method 1**: Call the three points $ABC$ in order. Then in coordinates the vectors $\overrightarrow{AB} = \langle 0,-1,2\rangle$ and $\overrightarrow{AC} = \langle -2,1,0\rangle$ lie on the plane, so their cross product

$$\vec{n} = \langle -2,-4,-2\rangle$$

is perpendicular to it. The plane therefore has an equation of the form $-2x-4y-2z = d$ for some $d$. Plugging in the coordinates of $C$ we see $d=-8$. Dividing by 2 and solving for $z$ we find the equation

$$\boxed{z = 4 - x - 2y}$$

**Method 2**: We would like to find coefficients $a,b,d$ so that the equation of the plane is $z = d - ax - by$. Plugging in the coordinates of $A,B,C$ we must then have :

$$\begin{cases} d-2a-b &= 0 \\ d-2a &= 2 \\ d-2b &= 0 \end{cases}.$$

Subtracting the first two equations we find $b=2$. Plugging this into the last equation gives $d=4$. Plugging this into the second equation gives $a=1$, so the equation is

$$\boxed{z = 4 - x - 2y}$$

(b) Find the point of intersection of the plane $p$ and the line $\ell$.

**Method 1**: Call the two points $D,E$ in order. Then in coordinates $\overrightarrow{DE} = \langle -1,1,2\rangle$, so we may parametrize the coordinates of the points of $\ell$ as

$$(2,0,0) + t\,\langle -1,1,2\rangle = (2-t,t,2t)\ .$$

We want $t$ so that this point lies on the plane, in other words such that

$$2t = 4 - (2-t) - 2t\,.$$

This simplies to $3t = 2$, that is $t = 2/3$ so the point of intersection is

$$\boxed{\left(\frac{4}{3},\frac{2}{3},\frac{4}{3}\right)}$$

**Method 2**: Since $\overrightarrow{DE} = \langle -1,1,2\rangle$, points $(x,y,z)$ on the line satisfy

$$\frac{x-2}{-1} = \frac{y}{1} = \frac{z}{2}\,.$$

Points on the plane satisfy $x+2y+z=4$, so the coordinates of the point of intersection satisfy the system of equations

$$\begin{cases} 2y = z \\ x = 2-y \\ z = 4-x-2y \end{cases}.$$

Substituting the first two equations into the third we find $2y = 4-(2-y)-2y$ so that $y = 2/3$. From this it follows that $x = 4/3 = z$ so we obtain the point

$$\boxed{\left(\frac{4}{3},\frac{2}{3},\frac{4}{3}\right)}$$

(c) Find the angle between the plane $p$ and the line $\ell$.

**Solution**: In part (a) we found that the vector $\langle 1,2,1\rangle$ is perpendicular to the plane. In part (b) we found that the vector $\langle -1,1,2\rangle$ lies along the line. The angle $\theta$ between these two vectors satisfies

$$\cos\theta = \frac{\langle 1,2,1\rangle\cdot\langle -1,1,2\rangle}{|\langle 1,2,1\rangle|\,|\langle -1,1,2\rangle|} = \frac{-1+2+2}{\sqrt{6}\cdot\sqrt{6}} = \frac{1}{2}\,.$$

This angle is therefore $\frac{\pi}{3}$. Dropping an altitude from a point on $\ell$ to $p$ we see that the angle between $\ell$ and $p$ is $\frac{\pi}{2}-\theta$, that is $\boxed{\frac{\pi}{6}}$.