Almost-Prime Times in Horospherical Flows

West Coast Dynamics Seminar

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May 28, 2020

Homogeneous Dynamics

- G, a Lie group
- $\Gamma \leq G$, a lattice (discrete, finite covolume subgroup)
- $X = \Gamma \backslash G$, space of interest
- $H \leq G$, a closed subgroup
- Dynamics: $H \curvearrowright X$ by right translations

Possible questions:

- Given $x \in X$, what does the orbit xH look like?
- What does a *typical* orbit look like?
- What H-invariant/ergodic measures are supported on this space?

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$G = \mathbb{R}^2, \ \Gamma = \mathbb{Z}^2, \ X = \mathbb{T}^2, \ H = \{tv \mid t \in \mathbb{R}\}$ for some $v \in \mathbb{R}^2$

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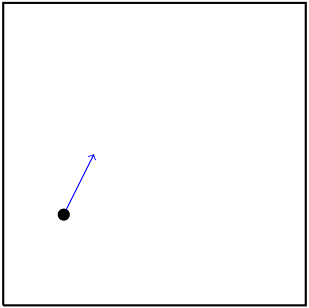

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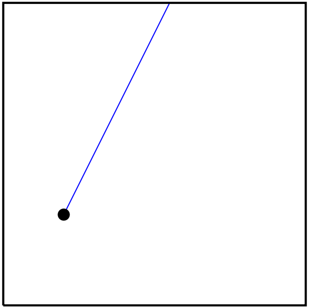

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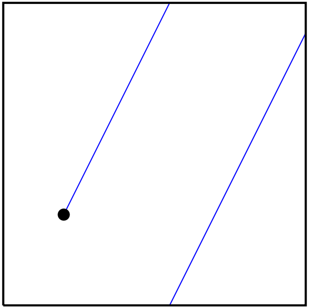

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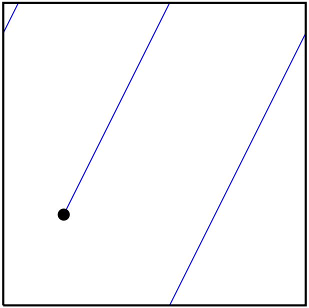

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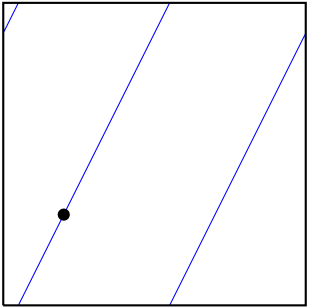

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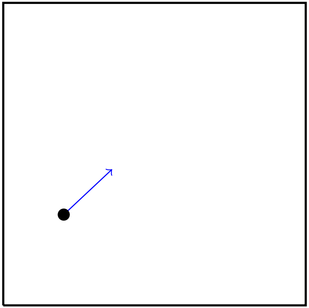

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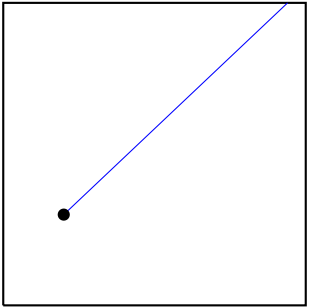

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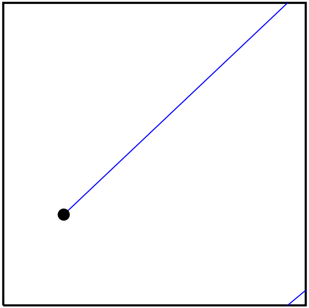

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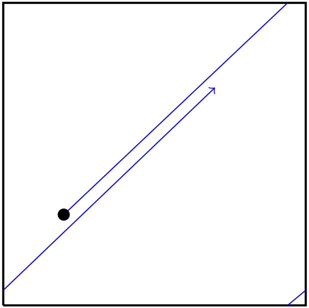

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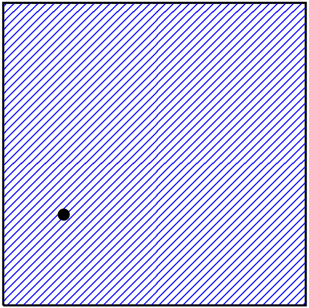

The Space of Lattices

- $G = \mathrm{SL}_2(\mathbb{R})$
- $\Gamma = \mathrm{SL}_2(\mathbb{Z})$
- $G \curvearrowright \mathbb{H}^2 := \{z = x + iy \in \mathbb{C} \,|\, y > 0\}$ by Möbius transformations:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

- $G \curvearrowright T^1\mathbb{H}^2$ by $g : (z, v) \mapsto (g(z), D_g v)$ with $\mathrm{Stab}_G(z) = \{\pm I\}$
- $\mathrm{PSL}_2(\mathbb{R}) \cong T^1\mathbb{H}^2$

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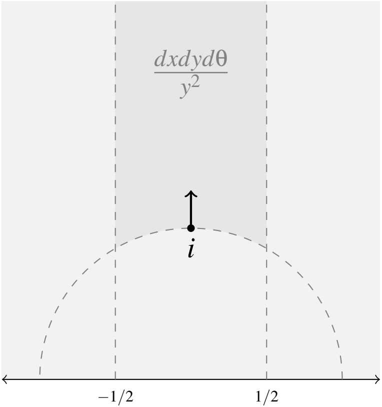
$\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R})$



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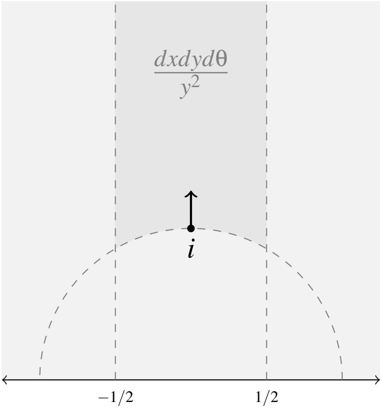

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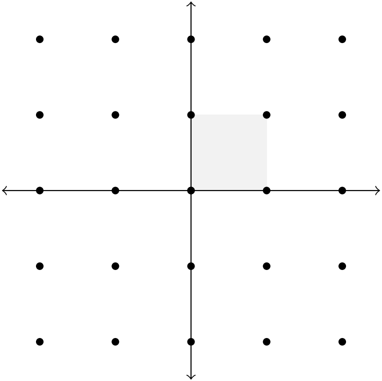

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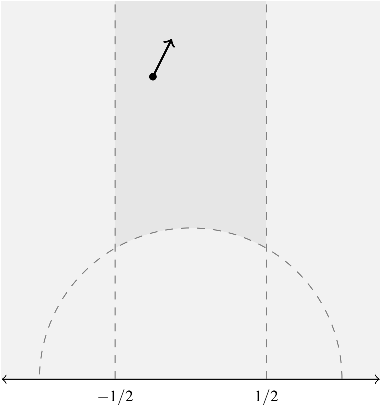

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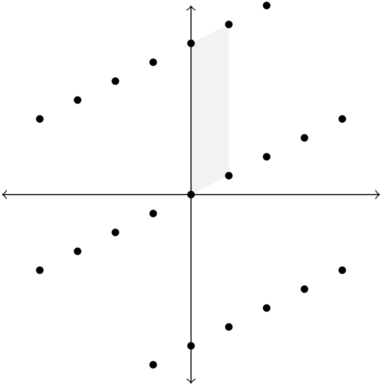

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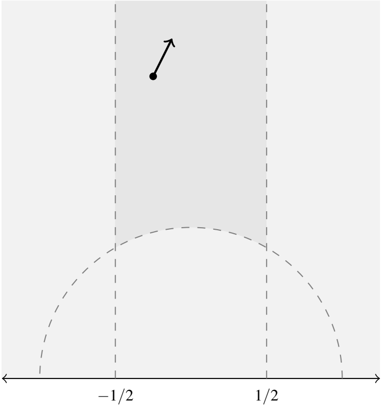

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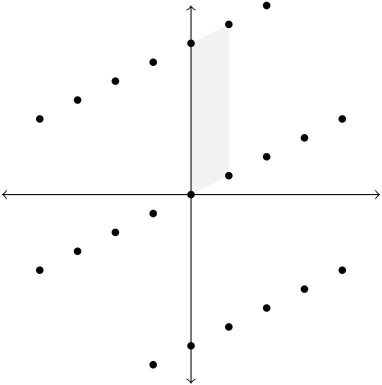

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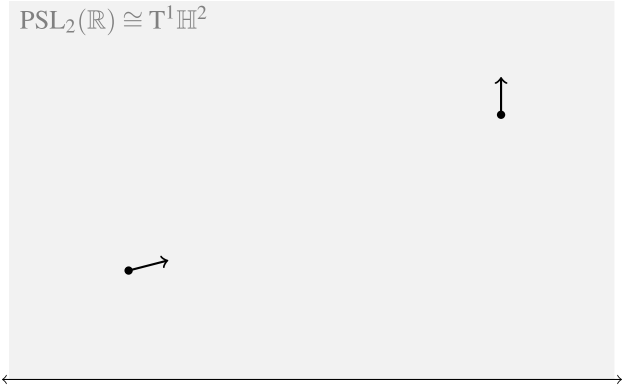
Subgroup Actions

geodesic flow

$$A = \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \right\}_{t \in \mathbb{R}}$$

horocycle flow

$$U = \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}_{t \in \mathbb{R}}$$



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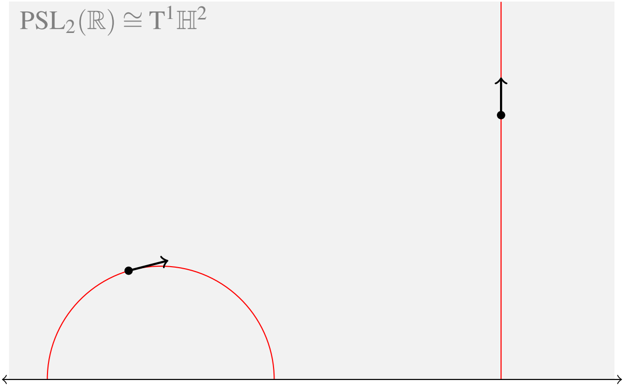

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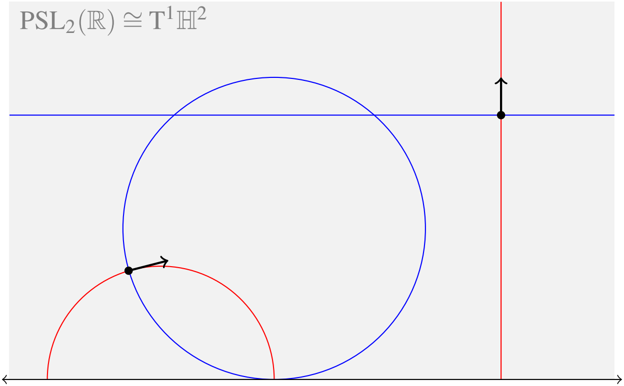

Horospherical Subgroups

Note: $a_t^{-1} u_s a_t = \begin{pmatrix} 1 & se^{-t} \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as $t \to \infty$

Definition

A subgroup $H \leq G$ is called *horospherical* if there exists $g \in G$ such that

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Fact: horospherical $\overset{\Longrightarrow}{\not\Longleftarrow}$ unipotent

Example (Heisenberg group)

$$\left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\} \text{ with respect to, e.g., } \begin{pmatrix} 2 & & \\ & 1 & \\ & & \frac{1}{2} \end{pmatrix}$$

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$$\left\{ \begin{pmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\} \text{ is NOT horospherical}$$

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Equidistribution

Roughly speaking, a subset of X *equidistributes* respect to a measure μ if it spends the expected amount of time in measurable subsets.

Example

A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ equidistributes with respect to μ if

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) \to \int_X f\,d\mu$$

for all $f \in C_c^\infty(X)$.

Say equidistribution is *effective* if the rate of convergence is known.

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Rigidity of Horospherical Actions

Theorem

Let $H \leq G$ be horospherical. For **any** $x \in X$*, there exists a closed, connected subgroup* $H \leq L \leq G$ *such that* $\overline{xH} = xL$ *and such that* xL *supports an* L*-invariant probability measure* μ_x *with respect to which the* H*-orbit of* x *equidistributes.*

- Hedlund, Furstenberg (SL_2)
- Burger (SL_2, Γ cocompact, effective w/ polynomial rate)
- Veech, Ellis-Perrizo (general horospherical, Γ cocompact)
- Margulis, Dani, Dani-Margulis (quantitative nondivergence)
- Dani (above theorem)
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Qualitative Equidistribution

Theorem (Dani)

For every $x = \Gamma g \in X$, either

$$\frac{1}{|B_T|}\int_{B_T} f(xu)du \xrightarrow[T\to\infty]{} \int_X f dm \quad \forall f \in C_c^\infty(X) \tag{1}$$

or there is a proper, nontrivial rational subspace $W \subset \mathbb{R}^n$ such that Wg is U-invariant.

- du Haar measure on U
- dm pushforward of Haar measure on G to X
- $B_T = a_{\log T} B_1^U a_{\log T}^{-1}$ expanding Følner sets
- If x satisfies (1), call it *generic*.
  (Birkhoff's Theorem $\implies$ almost every x is generic.)

Effective Equidistribution

Theorem (M.)

There exists $\gamma > 0$ such that for every $x = \Gamma g \in X$ and $T > R$ large enough, either:

$$\left| \frac{1}{|B_T|} \int_{B_T} f(xu)du - \int_X f dm \right| \ll_f R^{-\gamma} \quad \forall f \in C_c^\infty(X) \tag{2a}$$

or

$$\exists j \in \{1, \cdots, n-1\} \text{ and } w \in \Lambda^j(\mathbb{Z}^n) \setminus \{0\} \text{ such that } \|wg_0u\| < R \;\; \forall u \in B_T. \tag{2b}$$

- If x satisfies (2a) for fixed R and all large T, call it *R-generic*. Note: x is generic $\iff$ x is R-generic for all $R > 0$.
- Condition (2b) says that there is a rational subspace $W \in \mathbb{R}^n$ such that Wg is *R-almost* invariant when flowed up to time T.

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- Applications in number theory often require effective rates.

Möbius Disjointness

Recall: the Möbius function

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not squarefree} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \end{cases}$$

Conjecture (Sarnak)

$$\frac{1}{N}\sum_{n\leq N}\mu(n)f(T^n x) \to 0$$

for **any***:*

- *X compact metric space*
- $x \in X$
- *$T : X \to X$ continuous, zero topological entropy*
- $f \in C(X)$

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Möbius Disjointness

Partial results:

- Vinogradov/Davenport (circle rotations/translations on a compact group–effective)
- Green-Tao (nilflows—effective)
- Bourgain-Sarnak-Ziegler/Peckner (unipotent flows on homogeneous spaces—not effective)

Equidistribution of Primes

Conjecture (Margulis)

Let $\{u_t\}_{t\in\mathbb{R}}$ be a unipotent flow on a homogeneous space X. If $\{xu_t \mid t \in \mathbb{R}\}$ equidistributes in X, then so does $\{xu_p \mid p \text{ is prime}\}$.

Theorem (Bourgain)

For any measurable dynamical system $(X, \mathcal{B}, \mu, T)$ and $f \in L^2(X, \mu)$, the ergodic averages over primes

$$\frac{1}{\pi(N)} \sum_{\substack{p \leq N \\ p \text{ prime}}} f(T^p x)$$

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The Horocycle Flow at Almost-Prime Times

Definition

An integer is called *almost-prime* if it has fewer than a fixed number of prime factors.

Theorem (Sarnak-Ubis)

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The Horocycle Flow at Almost-Prime Times

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Horospherical Flows at Almost-Prime Times

Let $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma \leq G$ a lattice, and $u(\mathbf{t})$ a d-dimensional horospherical flow on $X = \Gamma \backslash G$. Define

$$\mathcal{A}_\ell(x) = \{xu(k_1, k_2, \cdots, k_d) \mid k_i \in \mathbb{Z} \text{ has fewer than } \ell \text{ prime factors}\}.$$

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1. *If Γ is cocompact, then there exists $\ell = \ell(n, d, \Gamma)$ such that for any $x \in X$, the set $\mathcal{A}_\ell(x)$ is dense in X.*
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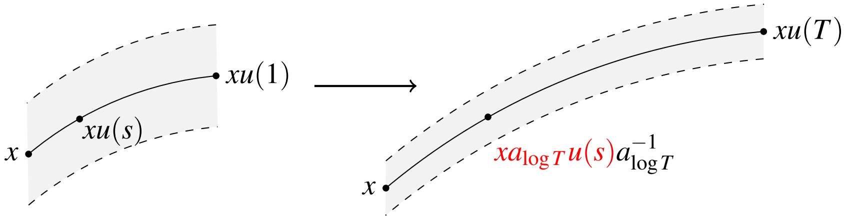
Questions?

Proof Idea:

1. Prove effective equidistribution of the continuous horospherical flow
2. Use this to prove effective equidistribution of arithmetic progressions of times
3. Apply sieve methods to deduce a statement about almost-primes

Effective Equidistribution of the Continuous Flow

Proof Idea: Margulis's thickening method



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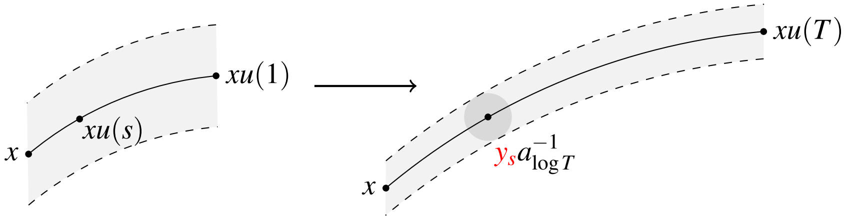

Effective Equidistribution of the Continuous Flow

Effective mixing of the A-action:

Theorem (Howe-Moore, Kleinbock-Margulis)

Let Γ be cocompact. There exists $\tilde{\gamma} > 0$ such that for any $x \in X$ and $f, g \in C_c^\infty(X)$,

$$\left| \int_X f(xa_t) g(x) dm - \int_X f dm \int_X g dm \right| \ll_{f,g} e^{-\tilde{\gamma} t}.$$

Note:

$$\frac{1}{|B_T|} \int_{B_T} f(xu) du = \int_{B_1} f(xa_{\log T} u a_{\log T}^{-1}) du$$
$$= \int_U \chi_{B_1}(u) f(xa_{\log T} u a_{\log T}^{-1}) du$$

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$$\int_U \chi_{B_1}(u) f(x a_{\log T} u a_{\log T}^{-1}) du$$

Problems:

1. χ_{B_1} not smooth
   - Convolve with a smooth approximation to the identity
2. Integral over U, not X
   - Thicken to get integral in G, project to X (need to make sure it injects)
3. Moving basepoint
   - Quantitative nondivergence (Dani-Margulis) $\implies$ can get a good radius of convergence for all but a small proportion of $u \in B_1$

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Effective Equidistribution of Arithmetic Progressions

Theorem (M.)

Let $u(t_1, \cdots, t_d)$ be an abelian horospherical flow. There exists $\beta > 0$ such that if $x \in X$ satisfies (2a) for $T > R$ large enough, then for any $1 \leq K \leq T$ we have

$$\left| \frac{K^d}{T^d} \sum_{\substack{\mathbf{k} \in \mathbb{Z}^d \\ K\mathbf{k} \in B_T}} f(xu(K\mathbf{k})) - \int_X f \, dm \right| \ll_f R^{-\beta} K^{d/(d+1)} \mathcal{S}(f).$$

Effective Equidistribution of Arithmetic Progressions

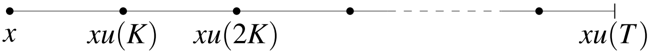
Proof Idea: Venkatesh's van der Corput method

For simplicity, assume $G = \mathrm{SL}_2(\mathbb{R})$, $\int f \, dm = 0$.

Let

$$E_{K,T}(f) = \frac{K}{T} \sum_{\substack{k \in \mathbb{Z} \\ 0 \le Kk < T}} f(xu(Kk))$$

be the average over the set:



Effective Equidistribution of Arithmetic Progressions

Define new function for $1 < H < T$:

$$f_H(x) = \frac{1}{H}\sum_{\ell=0}^{H} f(xu(K\ell))$$

Note: $E_{K,T}(f_H)$ is close to $E_{K,T}(f)$:

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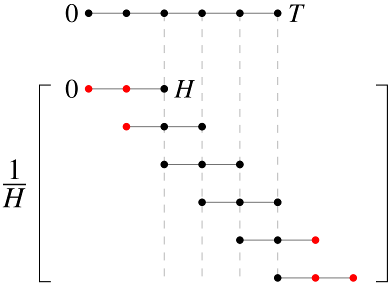

Effective Equidistribution of Arithmetic Progressions

Thicken the discrete set in U by $\delta > 0$:

Let $E_{K,T,\delta}$ be the ergodic average over this set.

Note: By uniform continuity, $E_{K,T,\delta}(f_H)$ is close to $E_{K,T}(f_H)$.

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$$E_{K,T,\delta}(f_H)^2 \ll \frac{K}{\delta H^2} \sum_{\ell_1=0}^{H} \sum_{\ell_2=0}^{H} \frac{1}{T} \int_0^T f(xu(s)u(K\ell_1)) f(xu(s)u(K\ell_2)) ds$$

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$\downarrow$ effective equidistribution

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$\downarrow$ bounds on matrix coefficients

$$\ll \frac{K}{\delta H^2} \sum_{\ell_1=0}^{H} \sum_{\ell_2=0}^{H} (1 + K|\ell_1 - \ell_2|)^{-a} \mathcal{S}(f)^2 + error$$

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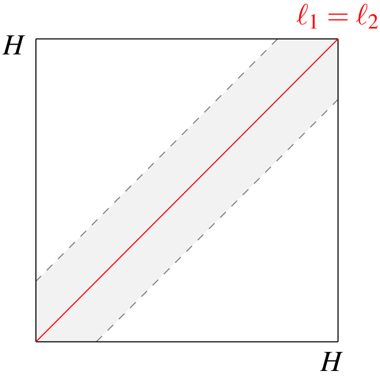

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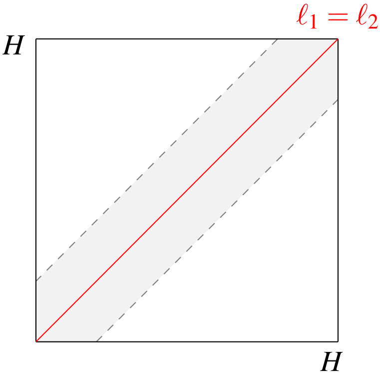

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Sieving

To sieve the orbits for almost-primes, need control over averages along arithmetic progressions—this is exactly what the last theorem tells us.

For $f \in C_c^\infty(X)$ and T large enough,

$$\frac{(\log T)^d}{T^d} \sum_{\substack{\mathbf{k} \in B_T \\ (k_1 \cdots k_d, P) = 1}} f(xu(\mathbf{k})) \asymp_\alpha \int f\, dm$$

where P is the product of primes less than T^α.

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