A $\mathbb{Z}^2$-Bratteli-Vershik model

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I will first describe the Bratteli-Vershik model for $\mathbb{Z}$-actions due to R. Herman, IFP, C. Skau, building heavily on the work of A. Vershik (quite successful) and then discuss such a model for $\mathbb{Z}^2$-actions (largely MIA) in some work in progress with T. Giordano and C. Skau.

X is a Cantor set (compact, metrizable, totally disconnected, no isolated points).

φ an action of $\mathbb{Z}^d, d = 1, 2$:

1. $n \in \mathbb{Z}^d$, $\varphi^n : X \to X$ is a homeomorphism,

2. $\varphi^m \circ \varphi^n = \varphi^{m+n}$, for all m, n.

3. φ is minimal if all orbits are dense.

An invariant

$C(X, \mathbb{Z}) = \{f : X \to \mathbb{Z} \mid f$ continuous $\}$ is a countable abelian group with point-wise addition.

$B(X, \varphi)$ generated by all functions of the form $f - f \circ \varphi^n$ with $f \in C(X, \mathbb{Z}), n \in \mathbb{Z}^d$.

$D(X, \varphi) = C(X, \mathbb{Z})/B(X, \varphi)$ (or $K^0(X, \varphi)$).

with order $D(X, \varphi)^+ = \{[f] \mid f \geq 0\}$. $[f]$ meaning the coset containing f.

This invariant contains information that classifies the system up to orbit equivalence.

Question: which countable, ordered, abelian groups can arise as the invariant of a Cantor minimal $\mathbb{Z}^d$-action ?

Today, I want to argue that the Bratteli-Vershik model is the answer to this question (at least one way): it takes a countable, ordered, abelian group and produces a Cantor minimal system.

This involves choices and in the $\mathbb{Z}$-case, the choices can be made so as to produce *any* Cantor minimal $\mathbb{Z}$-action. (We do not aim so high for $\mathbb{Z}^2$.)

Mini-Course on ordered abelian groups

$\mathbb{Z}^k$ has a *standard order*: $\mathbb{Z}^{k+}$ consists of n with $n_1, \ldots, n_k \geq 0$.

Given a sequence

$$\mathbb{Z}^{k_0} \stackrel{E_1}{\to} \mathbb{Z}^{k_1} \stackrel{E_2}{\to} \mathbb{Z}^{k_2} \stackrel{E_2}{\to} \ldots$$

E_j is a $k_j \times k_{j-1}$ matrix with non-negative integers entries we can produce an ordered abelian group.

$n \in \mathbb{Z}^{k_j}$ think of the sequence

$$(?, ?, \cdots, ?, n, E_{j+1}n, E_{j+2}E_{j+1}n, \ldots).$$

Two sequences are equal if they differ in finitely many entries. Obvious addition of sequences. A sequence is *positive* if all but finitely many terms are positive in $\mathbb{Z}^{k_j}$.

Example 1:

$$\mathbb{Z} \xrightarrow{[1]} \mathbb{Z} \xrightarrow{[2]} \mathbb{Z} \xrightarrow{[3]} \cdots$$

and the limit group is the rational numbers $\mathbb{Q}$.

Example 2:

$$\mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \cdots$$

$G = \mathbb{Z}^2$,

$G^+ = \{(n_1, n_2) \mid n_1\gamma + n_2 \geq 0\}$ where γ is the golden mean.

Theorem 1 (Effros-Handelman-Shen)**.**

Let (G, G^+) *be a countable ordered abelian group. It is an inductive limit of* $(\mathbb{Z}^{k_j}, \mathbb{Z}^{k_j}{+})$ *as above if and only if*

1. *it is unperforated:* a *in* G*,* $n \geq 1$ *with* na *in* G^+ *implies* a *in* G^+*,*

2. *it has Riesz interpolation: for* $a, b \leq c, d$*, there is* e *with* $a, b \leq e \leq c, d$*.*

Such groups are called *dimension groups*.

Corollary 2. *Let* (G, G^+) *be a countable ordered abelian group. TFAE:*

1. *It is an inductive limit of* $(\mathbb{Z}^{k_j}, \mathbb{Z}^{k_j +})$ *as above with matrices* E_j *which have positive entries.*

2. *It is unperforated, has Riesz interpolation and is simple: for any* $a \neq 0$ *in* G^+ *and* b *in* G*, there in* n *with* $na \geq b$*.*

3.

Corollary 3. *Let* (G, G^+) *be a countable ordered abelian group. TFAE:*

1. *It is an inductive limit of* $(\mathbb{Z}^{k_j}, \mathbb{Z}^{k_j +})$ *as above with matrices* E_j *which have positive entries.*

2. *It is unperforated, has Riesz interpolation and is simple: for any* $a \neq 0$ *in* G^+ *and* b *in* G, *there in* n *with* $na \geq b$.

3. *There is a minimal action* φ *of* $\mathbb{Z}$ *on a compact, totally disconnected metric space* X *with*

$$(G, G^+) \cong (D(X, \varphi), D(X, \varphi)^+).$$

The Bratteli-Vershik model is the proof of (1) implies (3).

First convert groups and matrices to vertices and edges: $\mathbb{Z}^k$ becomes k-vertices, $\{1, 2, 3, \ldots, k\}$.

E and $k' \times k$ matrix becomes the edges in a bipartite graph from $\{1, 2, 3, \ldots, k\}$ to $\{1, 2, 3, \ldots, k'\}$: $E_{i,j}$ is the number of edges from j to i.

The result is called a Bratteli diagram.

To dynamics:

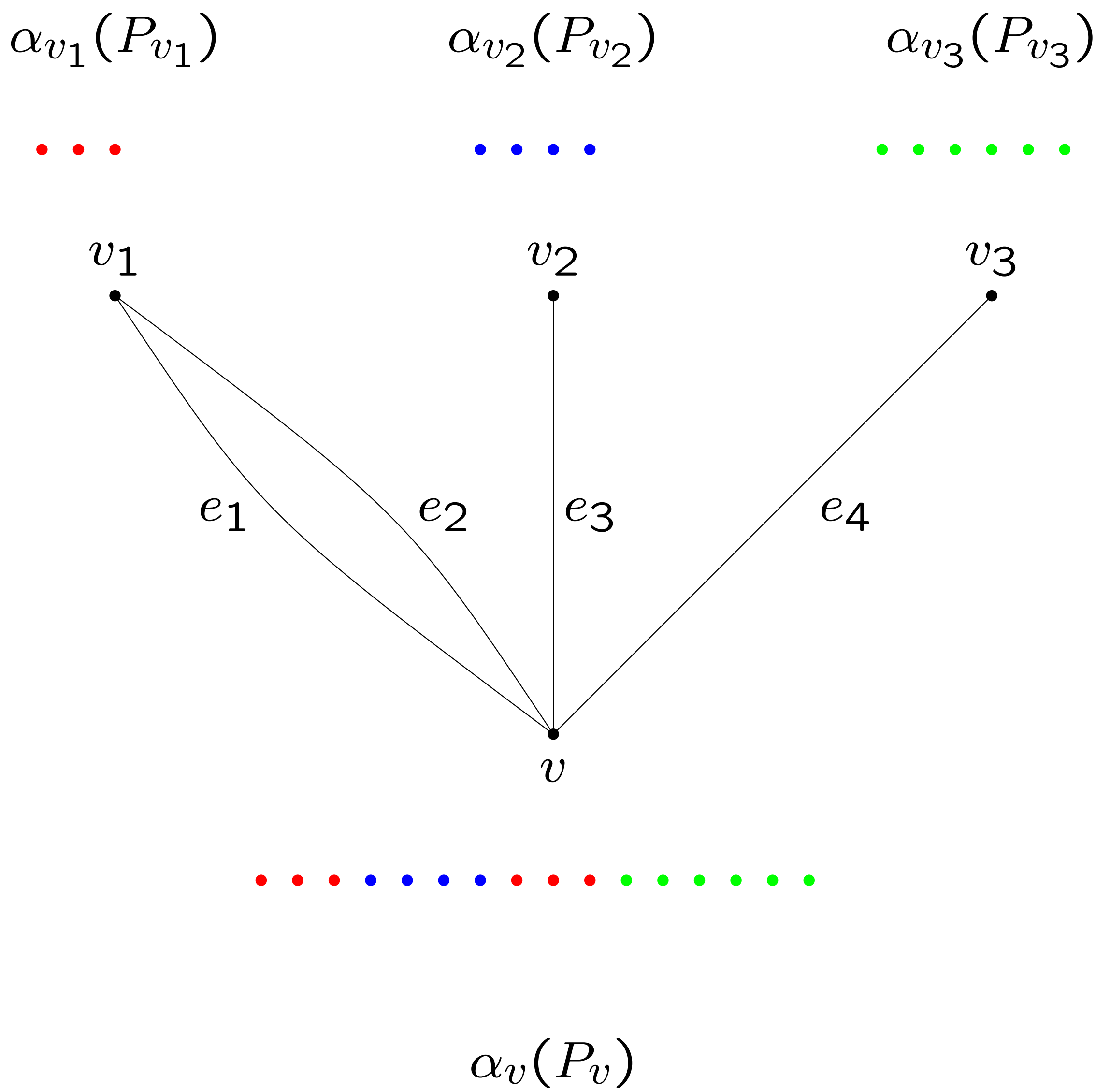
X is the set of infinite paths in the diagram, starting from level 0.

For φ: for each vertex v, let P_v be the set of all paths from level 0 to v. We look for an injection

$$\alpha_v : P_v \to \mathbb{Z}$$

with a 'nice' image. 'Nice' might mean something like a Følner set, but really just means interval, in a sense appropriate for $\mathbb{Z}$.

The key point is to make the maps α_v for v at level $n+1$ compatible in a sense with those from level n



$$P_v = P_{v_1}e_1 \cup P_{v_1}e_2 \cup P_{v_2}e_3 \cup P_{v_3}e_4.$$

$$\alpha_v(pe_1) = \alpha_{v_1}(p) + t_{e_1}.$$

The map φ is obtain in a limiting process

$$\alpha_v(\varphi^m(x)_1, \ldots, \varphi^m(x)_n) = \alpha_v(x_1, \ldots, x_n) + m,$$

with $v = t(x_n)$ and $m \in \mathbb{Z}$.

There are obvious problems at the boundaries of the regions and care must be taken to ensure:

1. φ is a homeomorphism,

2. $(D(X, \varphi), D(X, \varphi)^+) \cong (G, G^+)$.

This needs the hypothesis that the matrices have no zero entries = the diagram has full edge connections.

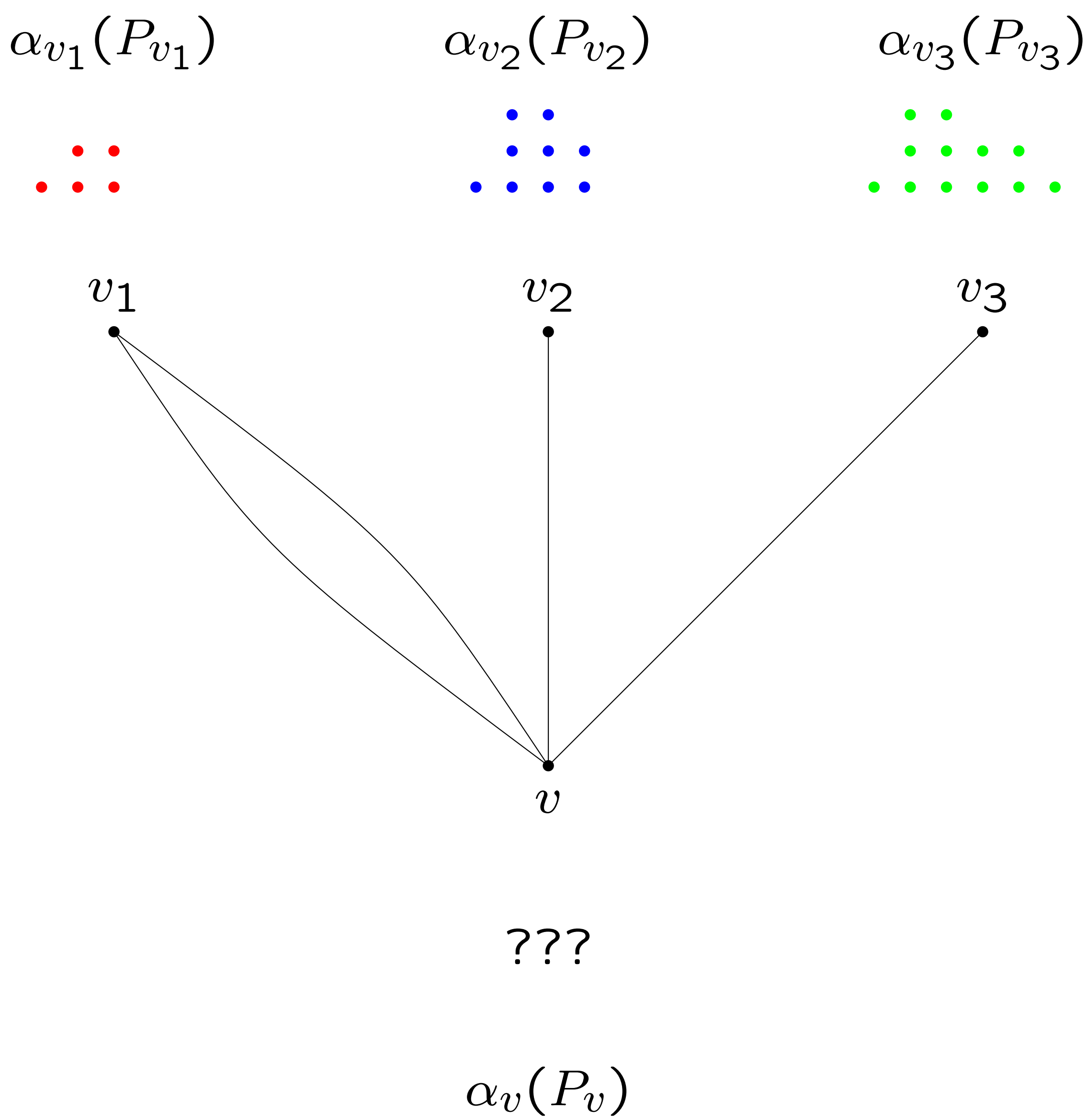
Can this be done for $\mathbb{Z}^2$-actions?

Everything goes well until we get to the dynamics: our injections

$$\alpha_v : P_v \to \mathbb{Z}^2$$

with 'nice' images.

Now, 'nice' isn't so clear. And what is worse is the question of fitting them together:


$\alpha_{v_1}(P_{v_1})$
$\alpha_{v_2}(P_{v_2})$
$\alpha_{v_3}(P_{v_3})$
v_1
v_2
v_3
v
???
$\alpha_v(P_v)$

Starting over to ask: can this be done for $\mathbb{Z}^2$-actions?, it is useful to see where our invariant $D(X, \varphi)$ came from.

$\mathbb{Z}^d$ is an abelian group acting on the abelian group $C(X, \mathbb{Z})$ by automorphisms. Therefore, there is *group cohomology of $\mathbb{Z}^d$ with coefficients in $C(X, \mathbb{Z})$*:

$$H^k(\mathbb{Z}^d, C(X, \mathbb{Z})), k = 0, 1, 2, \ldots$$

which we prefer to write as $H^k(X, \mathbb{Z}^d, \varphi)$.

Case $d = 1$ and φ minimal.

$$H^0(X, \mathbb{Z}, \varphi) \cong \mathbb{Z},$$

(not interesting)

$$H^1(X, \mathbb{Z}, \varphi) \cong D(X, \varphi),$$

(our invariant!)

$$H^k(X, \mathbb{Z}, \varphi) = 0, k \geq 2,$$

(even less interesting).

Case $d = 2$ and φ minimal.

$$H^0(X, \mathbb{Z}^2, \varphi) \cong \mathbb{Z},$$

(not interesting)

$$H^2(X, \mathbb{Z}^2, \varphi) \cong D(X, \varphi),$$

(our invariant!)

$$H^k(X, \mathbb{Z}^2, \varphi) = 0, k \geq 3,$$

(even less interesting).

But what about H^1?

One of our main points here is that H^1 is (1) much more interesting than the invariant we've been focused on and (2) under-appreciated.

To give some evidence for (1), if one restricts to the class of $\mathbb{Z}^2$-odometers, H^1 is a complete invariant for topological conjugacy while H^2 is a complete invariant for orbit equivalence. The picture is muddied by the fact that, for $\mathbb{Z}$-odometers, the two notions coincide.

$H^1(X, \mathbb{Z}^2, \varphi)$

Look at all $\theta : X \times \mathbb{Z}^2 \to \mathbb{Z}$ which are continuous and satisfy

$$\theta(x, m+n) = \theta(x, m) + \theta(\varphi^m(x), n),$$

for all $x \in X, m, n \in \mathbb{Z}^2$.

Notice that this includes $Hom(\mathbb{Z}^2, \mathbb{Z})$ as functions constant in x. More generally, if μ is a φ-invariant probability measure on X, then

$$\int \theta(x, \cdot) d\mu(x)$$

is in $Hom(\mathbb{Z}^2, \mathbb{R})$.

If $h : X \to \mathbb{Z}$ is continuous, then

$$bh(x, n) = h(x) - h(\varphi^n(x)),$$

is such a function.

$H^1 = \{\theta\}/\{bh\} = \ker(b)/b(C(X, \mathbb{Z}))$.

For an invariant probability measure μ, we define a group homomorphism

$$\tau_\mu : H^1 \to \mathbb{R}^2$$

by

$$\tau_\mu(\theta) = \left(\int \theta(x,(1,0))d\mu(x), \int \theta(x,(0,1))d\mu(x) \right)$$

which is a homomorphism with $\mathbb{Z}^2 \subseteq \tau_\mu(H^1)$.

One can also show that $H^1(X, \mathbb{Z}^2, \varphi)$ is torsion-free.

Our new question is: what groups can arise as $H^1(X, \mathbb{Z}^2, \varphi)$? and the Bratteli-Vershik model is the answer.

We will see the exact spot in the proof where H^1 (instead of H^2) being our target group changes what we were doing before.

Theorem 4 (Giordano-P-Skau). *Let H be a torsion-free, countable abelian group and $\tau : H \to \mathbb{R}^2$ be a homomorphism such that*

1. $\mathbb{Z}^2 \subseteq \tau(H)$,

2. $\tau(H) \subseteq \mathbb{R}^2$ *is dense in* $\mathbb{R}^2$.

Then there is a minimal action, φ, of $\mathbb{Z}^2$ on the Cantor set, X, with unique invariant measure μ such that

1. $H^1(X, \mathbb{Z}^2, \varphi) \cong H$,

2. $\tau_\mu = \tau$ *(with the identification above).*

On the hypothesis that $\tau(H) \subseteq \mathbb{R}^2$ is dense in $\mathbb{R}^2$.

This is true for many standard examples and we conjectured that it would always be true for minimal systems. However, Alex Clark and Lorenzo Sadun have an example of a minimal $(X, \mathbb{Z}^2, \varphi)$ with X Cantor where

$$\tau_\mu : H^1(X, \mathbb{Z}^2, \varphi) \to \mathbb{Z}^2$$

is an isomorphism.

So our Bratteli-Vershik model cannot produce this example (as it stands).

Begin with $\tau : H \to \mathbb{R}^2$ and produce X, φ.

Step 1:

Proposition 5. *With*

$$H^+ = \{0, h \in H \mid \tau(h) \in (0, \infty)^2\}$$

(H, H^+) *is unperforated, has Riesz interpolation and is simple.*

Riesz interpolation needs $\tau(H)$ dense in $\mathbb{R}^2$. We need the strict first quadrant to get simple.

Step 2: Effros-Handelman-Shen writes H, H^+ as an inductive limit of $\mathbb{Z}^k, \mathbb{Z}^{k+}$ and so provides us with a Bratteli diagram with full edge connections.

Step 3: The fact that $\mathbb{Z}^2 \subseteq \tau(H)$ means that H has elements that map to $(1, 0)$ and $(0, 1)$. These aren't positive, but almost are and their sum is actually quite large in this group. The group has a very special structure and we can arrange it as an inductive limit

$$\mathbb{Z}^{2k_0} \xrightarrow{E_1} \mathbb{Z}^{2k_1} \xrightarrow{E_2} \mathbb{Z}^{2k_2} \xrightarrow{E_2} \ldots$$

with

$$E_j = \left[\begin{array}{cc} \text{large} & \text{small} \\ \text{small} & \text{large} \end{array} \right].$$

Step 4: we again look for embeddings: for any vertex v and P_v all paths to v,

$$\alpha_v : P_v \to \mathbb{Z}^2$$

but the images will not be blobs/Følner sets, but *paths* in $\mathbb{Z}^2$.

Why paths? Because we are trying to build our dynamics to have H^1 given by the Bratteli diagram and so we are trying to build its "1-skeleton".

Now the issue making the choices between adjacent levels coherent means that instead of putting blobs together, we are putting paths together and that is easy: concatenate.

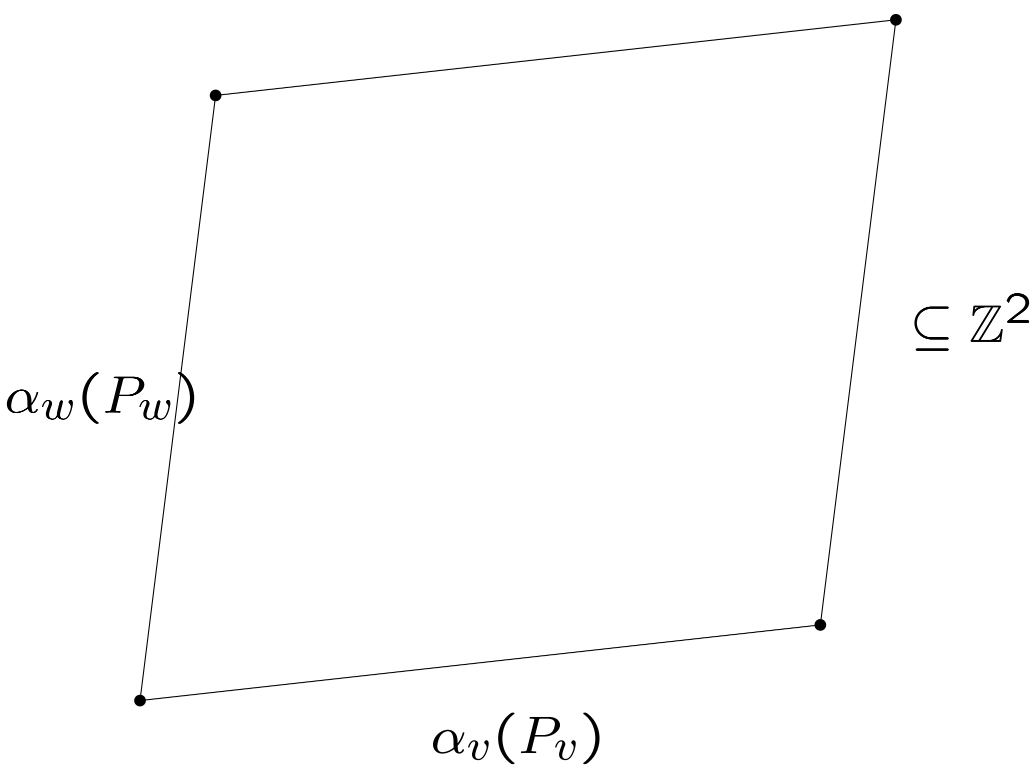
Recall that our $2k_j \times 2k_{j-1}$-matrix looks like

$$E_j = \begin{bmatrix} \text{large} & \text{small} \\ \text{small} & \text{large} \end{bmatrix}.$$

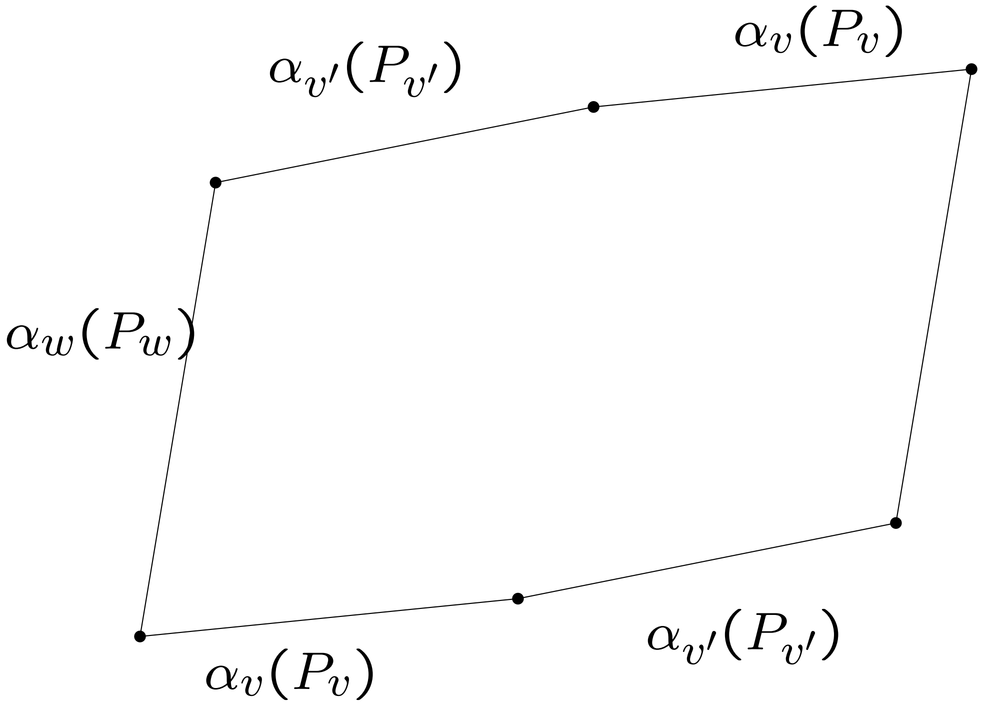
Also recall that $V_j = \{1, \ldots, 2k_j\}$. For vertices $1 \leq v \leq k_j$, $\alpha_v(P_v)$ looks mostly horizontal, while for $k+1 \leq w \leq 2k_j$, $\alpha_w(P_w)$ looks mostly vertical.

Going to level j, from $j-1$, the matrix says that, for the first set of vertices, we should concatenate a large number of mostly horizontal paths with a small number of mostly vertical ones and the result will be even more horizontal.

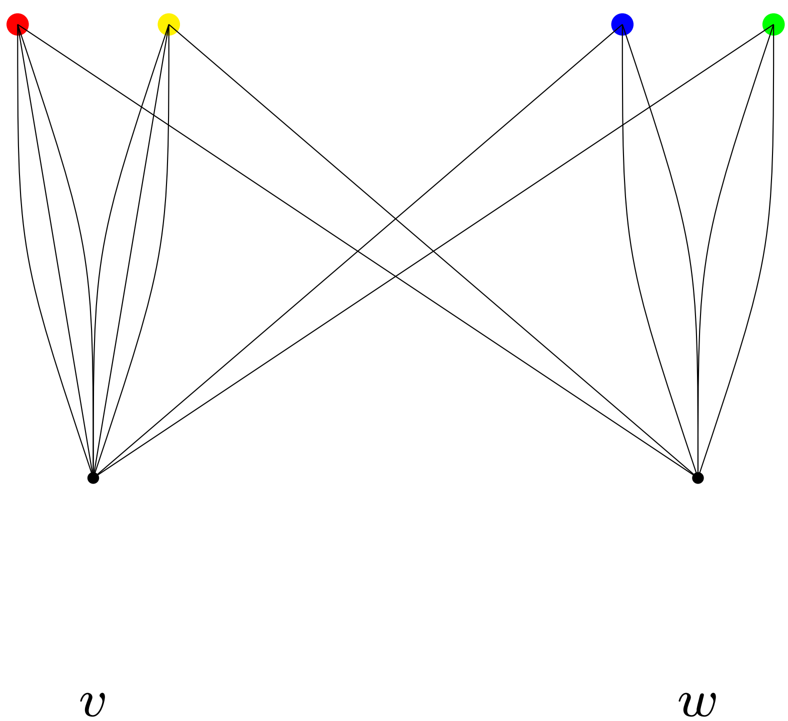
For each pair $1 \le v \le k_j < w \le 2k_j$, we form a region $R(v; w)$ as follows.



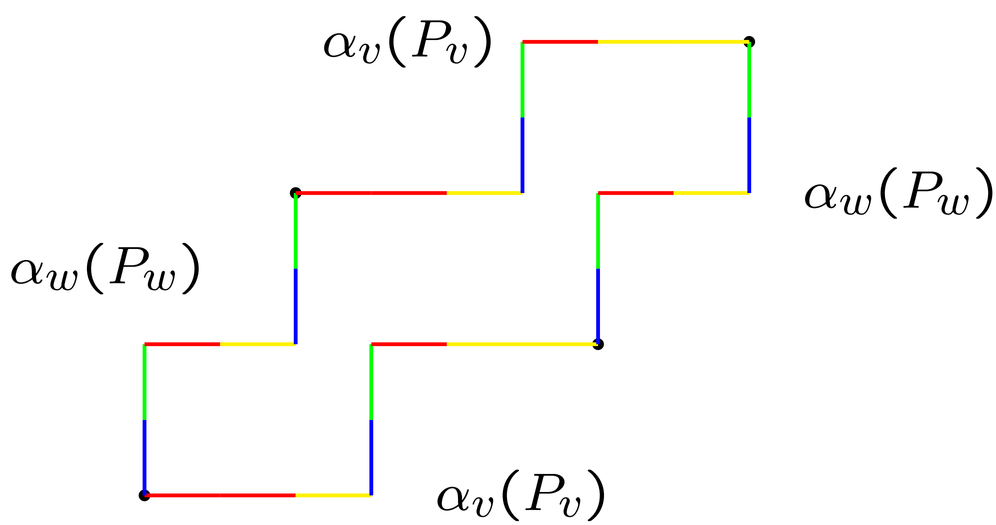
In addition, for $1 \leq v \neq v' \leq k < w \leq 2k$, we also need $H(v, v'; w)$:

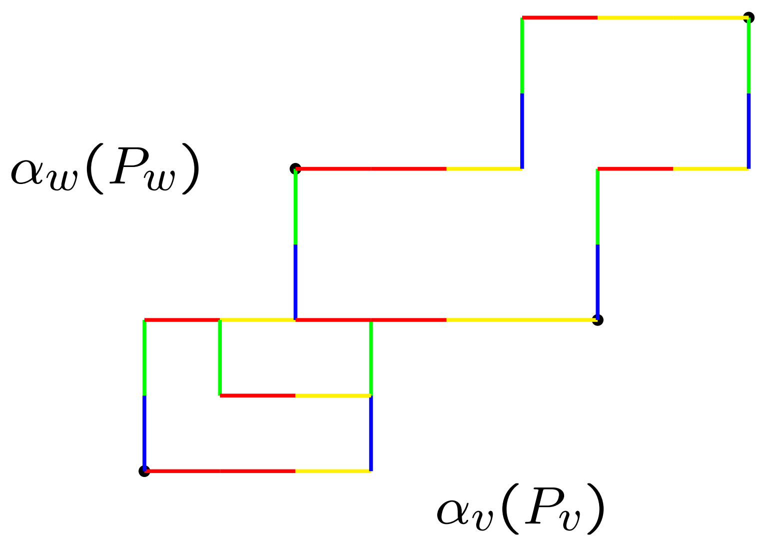


and also $H(v; w, w')$ for $1 \leq v \leq k < w \neq w' \leq 2k$


v
w

$R(v; w)$





$\alpha_w(P_w)$
$\alpha_v(P_v)$