HOMEWORK 6: Math 265 Leah Keshet Due in Class on Nov 3

Problem 1: Use the Laplace Transform table and properties of the Laplace transform to find the Laplace transforms of the following functions (You should not need to use any integration here):

(a) $f(t) = \sinh(t)$. (Comment: the "hyperbolic" trig functions are defined as follows:

$\sinh(t) = \frac{1}{2}(e^{at} - e^{-at})$ and $\cosh(t) = \frac{1}{2}(e^{at} + e^{-at})$.

You will find that their transforms happen to look somewhat similar to the transforms of the "usual" trig functions $\sin(t)$ and $\cos(t)$.

(b) $f(t) = t^2 e^{at}$.

(c) $f(t) = e^{at}(H(t-1) - H(t-2))$ where $H(t)$ is the Heaviside step function. (Note: in the notation of your book, this function can be written as $f(t) = e^{at}(u_1(t) - u_2(t))$.

Problem 2: Consider the discontinuous function shown in the Figure 1.

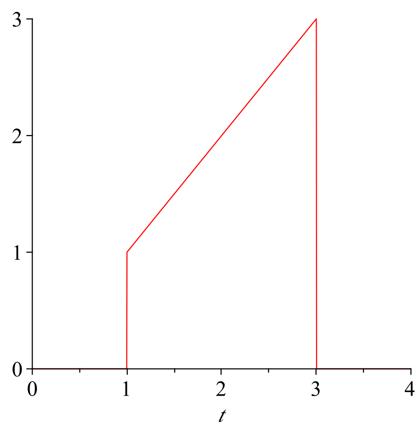

Figure 1:

(a) Express this function in terms of step function(s) multiplied by the appropriate terms.

(b) Find the Laplace Transform of this function.

(c) Consider a function of period $T = 4$ for which this is one "cycle". Make a rough sketch of what the periodic function looks like over 2-3 cycles.

(d) What is the Laplace transform of this periodic function?

Problem 3: Find the inverse Laplace transform in each case. (Note: factoring or completing the square will be handy. Keep a copy of the Laplace transform table and the facts about those Laplace transforms nearby to help you out.)

(a) $F(s) = \dfrac{1}{(s-1)^3} + \dfrac{1}{s^2+2s-8}$

(b) $F(s) = \dfrac{e^{-\pi s/2}}{s^2+9}$

(c) $F(s) = \dfrac{6}{s^3-9s}$

Problem 4: Solve the following initial value problems for $y(x)$:

(a) $y'' + y = H(t-3\pi), \quad y(0) = 1, y'(0) = 0.$

(b) $y'' - 2y' + y = e^t, \quad y(0) = 0, y'(0) = 1$

(c) $y'' + 4y' + 5y = \delta(t-2\pi), \quad y(0) = 0, y'(0) = 0$ where $\delta(t)$ is the Dirac delta function.

Problem 5:

(a) Use the fact that $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$ to find the Laplace transforms of the functions $t\sin(\omega t)$ and $t\cos(\omega t)$.

(b) Use Laplace transforms to solve the initial value problem
$y'' + 25y = \cos(5t), y(0) = 0, y'(0) = 1.$
(Hint: in the inversion step, you will find the result of part (a) useful).

Problem 6: Consider the problem we saw before in HW2: A patient is in the hospital on intravenous medication and $I(t)$ is the rate at which medication is infused (injected into the patient). We saw that $c(t)$, the drug in the bloodstream satisfies the initial value problem

$$\frac{dc}{dt} = I(t) - rc, \quad c(0) = 0.$$

where $r > 0$ is the rate of breakdown of the drug in the liver. Suppose the infusion rate is turned "on" at $t = 0$ and "off" at $t = 1$, so that $I(t) = 1$ for $0 \le t \le 1$ and 0 otherwise.

(a) Express $I(t)$ in terms of step function(s).

(b) Solve the above equation using the Laplace transform method with the function $I(t)$ given in part (a).