HOMEWORK 5: MATH 265 Due in class on Oct 27

Problem 1: Improper integrals.

(a) Consider the integral $I = \int_1^\infty \frac{1}{t^p}\, dt$. Show that this improper integral converges for $p > 1$ and find its value. Show that it diverges for $p = 1$. What happens when $p < 1$?

(b) Consider the integral $I = \int_0^\infty e^{-(5-k)t}\, dt$. Does this integral converge? To what value and under what condition(s) if any.

(c) Consider the integral $I = \int_0^\infty \frac{x}{10^5}\, dx$. Determine whether this integral converges or diverges and explain why.

Problem 2: A *Comparison Theorem* states the following two facts:
(1) If $0 \le g(x) \le f(x)$ and $I_1 = \int_0^\infty f(x)\, dx$ converges, then $I_2 = \int_0^\infty g(x)\, dx$ also converges.
(2) If $0 \le g(x) \le f(x)$ and $I_2 = \int_0^\infty g(x)\, dx$ diverges, then $I_1 = \int_0^\infty f(x)\, dx$ also diverges.

Use these theorems, together with the results of Problem 1 to determine which of the following integrals converges and briefly explain your answer.

(a) $I = \int_1^\infty \frac{1}{1+x^3}\, dx$

(b) $I = \int_1^\infty \frac{1}{\sqrt{x-0.5}}\, dx$

Problem 3: Show the detailed steps in computing the Laplace transform of the following functions. (You can check your answer using a table of Laplace transforms, but you are required to actually do the integration to verify your answers.)

(a) $f(t) = Kt$ where K is a constant.

(b) $f(t) = e^{-t/\tau}$ where τ is a constant.

(c) $f(t) = \sin(at)$ and $f(t) = \cos(at)$ (Note: you will find that this double integration by parts will allow you to find both of these "together".)

(d) Let $f(t), g(t)$ be acceptable functions. Verify that $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$, i.e. show (using the appropriate integrals) that the Laplace transform is linear.

Problem 4: Suppose that $F(s) = \mathcal{L}\{f(t)\}$ is the Laplace transform of $f(t)$. In class we showed that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

(a) Show that $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$. (**Note: sorry that the notes in class had a confusion in the last two terms! please correct it.. the scanned lecture notes are correct.**)

(b) Let $f^{(n)}(t)$ be the nth derivative of the function $f(t)$ and let $M_n = \mathcal{L}\{f^{(n)}(t)\}$ be its Laplace transform. Show that $M_n = sM_{n-1} - f^{(n-1)}(0)$.

(c) **Bonus**: Use the above *recursion relation* between M_n and M_{n-1} to arrive at the general "formula" for $\mathcal{L}\{f^{(n)}(t)\}$.

Problem 5:

(a) Find the Laplace transform for the following (discontinuous) function:

$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$$

(b) Prove the following theorem about shift (translation) of the Laplace transform:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

(c) Use your result in part (b) to find the inverse Laplace transform for the $F(s) = \dfrac{1}{6(s-1)^3}$.

Problem 6: Find the solution to the following ODEs using the Laplace transform

(a) $y'' - 3y' + 2y = 12e^{4t}$ with $y(0) = 1, y'(0) = 0$.

(b) $y'' + y' - 2y = 4e^t + 1$ with $y(0) = 1, y'(0) = 0$. (Note: this is the example we discussed in class where there is a "hard way" to do the algebra and a somewhat easier way.)

(c) $y'' + 4y' - 5y = te^t$ with $y(0) = 1, y'(0) = 0$. (Hint: the shift theorem will be useful. There will be some algebra in this one.)