HOMEWORK 3 (FINAL VERSION): Math 265 Leah Keshet
This homework will not be collected, as we have a test on Oct 6. However, it is
important that you know how to do these problems. Be sure and do them well
ahead of the midterm. Solutions will be posted on the weekend of Oct 2. Note:
Problems 1-4 and their solutions are from a previous year. I thank Prof Daniel
Coombs for making these available to us for your practice and learning.

Problem 1: Solve the following initial value problems for $y(x)$:
(a) $y'' - 4y' - 5y = 0$, $y(-1) = 3$, $y'(-1) = 9$.
(b) $y''' + 2y'' - 5y' - 6y = 0$, $y(0) = 2$, $y'(0) = 6$, $y''(0) = 0$.
(Hint: this is a linear equation whose characteristic equation is cubic. Recall that for a cubic equation
$r^3 + ar^2 + br + c = 0$ with roots r_1, r_2, r_3, it is true that $c = r_1 r_2 r_3$.)
(c) $y'' + y = 2e^{-x}$, $y(0) = 0$, $y'(0) = 0$.
(d) $y'' + 2y' + y = x^2 + 1 - e^x$, $y(0) = 0$, $y'(0) = 2$.
(e) $y'' - 2y' + y = 8e^t$, $y(0) = 3$, $y'(0) = 2$.
(f) $y'' + 2y' + 2y = 5\cos(2x)$, $y(\pi) = -1/2$, $y'(\pi) = 1$.

Problem 2: The suspension in a car can be modeled as a vibrating spring with damping due to the shock
absorbers. This leads to the equation for the vertical displacement $x(t)$ at time t,

$$mx''(t) + bx'(t) + kx(t) = 0,$$

where m is the mass of the car, b is the damping constant of the shocks, and k is the spring constant. If the
mass m of the car is 1000kg and the spring constant k is 3000kg/s^2, determine the minimum value for the
damping constant b in kilograms per seconds that will provide a smooth, *oscillation-free* ride. If we replace
the springs with heavy-duty ones having twice the spring constant k, how will this minumum change?

Problem 3: A vibrating spring *without* damping can be modeled by the initial value problem:

$$my''(t) + ky(t) = 0 \quad y(0) = y_0, \quad y'(0) = y_1$$

for m the mass of the spring and k is the spring constant.
(a) If $m = 10$kg, $k = 250$kg/s^2, $y_0 = 0.3$m, and $y_1 = -0.1$m/s, find the equation of motion $y(t)$ for this
undamped vibrating spring.
(b) What is the frequency of oscillation of this spring system?

Problem 4: A vibrating spring *with* damping can be modeled by the initial value problem:

$$my''(t) + by'(t) + ky(t) = 0 \quad y(0) = y_0, \quad y'(0) = y_1$$

for m the mass of the spring, k is the spring constant, and b the damping constant.

(a) Using the same values for m, k, y_0, and y_1 as in Problem 3, now with $b = 60\text{kg/s}$, find the equation of motion $y(t)$ for this damped vibrating spring.

(b) What is the frequency of oscillation of this spring system?

(c) Compare the results of problems 3 and 4 and determine what effect the damping has on the frequency of oscillation. What other effects does it have on the solution? What is the long-time behaviour of the solution (behaviour of the solution as $t \to \infty$)?

Problem 5 Consider the *nonhomogeneous* second order ODE

$$y'' - 2y' - 3y = 2e^{-t}.$$

The general solution of this equation is $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y_p(t)$ where $y_1(t), y_2(t)$ are a fundamental set of solutions to the corresponding homogeneous ODE and $Y_p(t)$ is a particular solution to the nonhomogeneous ODE.

(a) Suppose we "guess" a form for the particular solution as $Y_p(t) = Ae^{-t}$ (since this is similar to the form of the time-dependent forcing term.) Plug this function into the ODE and show that you arrive at a contradiction. Why does this happen?

(b) Now revise your guess to the form $Y_p(t) = Ate^{-t}$. Show that this works, find the value of A, and thereby also find the general solution to the nonhomogeneous ODE.
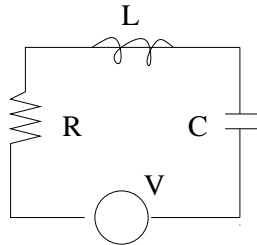
Problem 6

Figure 1: Circuit for problems 6 and 7

Consider a circuit with a resistor, inductor, and capacitor in series (Fig 1), and suppose this is connected to a constant voltage V. Recall that the ODE satisfied by the charge $q(t)$ across the capacitor in such a circuit is

$$Lq'' + Rq' + (1/C)q = V.$$

Also recall that the current $I(t)$ in the circuit is related to the charge $q(t)$ by $I = q'(t)$.

(a) Use the above information to find the differential equation satisfied by the current $I(t)$.

(b) Consider the (unrealistic) case that the resistance in $R = 0$ in this circuit. Determine the behaviour of $I(t)$, i.e. find the general solution to the equation you found in part (a). What is the frequency of the oscillation?

(c) Now suppose that R is gradually increased. At what value of R will there be *no oscillation*? Sketch the behaviour of $I(t)$ for values of R below and above that critical value.

(d) Someone has set up the circuit with ($R \neq 0$) so that there is initially a charge on the capacitor when a switch is closed so that at time $t = 0$, $q(0) = q_0$ and $I(0) = I_0$ are known. Find $I(t)$ using these initial conditions.

Problem 7 Consider the circuit shown in Fig 1 and assume that $V = 0$ and a switch is closed at $t = 0$. In this circuit, the inductance is $L = 0.05$ Henrys, Capacitance is $C = 2 \times 10^{-4}$ Farads and the resistance is $R = 10\Omega$. The initial charge on the capacitor at time $t = 0$ is 2 coulombs. Determine the current $I(t)$ for $t \geq 0$.