Sept 29 Characteristic Eqn has repeated roots

<u>Example</u> Consider the ODE

$$\circledast \quad y'' + 2y' + y = 0 \qquad \text{with initial cond's } y(0)=1,\ y'(0)=3$$

then char. eqn $r^2 + 2r + 1 = 0$ has $r = -1, -1$ (repeated roots)

$\Rightarrow$ $y_1(t) = e^{-t}$ is one solution, but what should we use for y_2?

No good: $y_2(t) = e^{-t}$ $\leftarrow$ need a distinct fn. such that $W(y_1, y_2) \neq 0$ (wronskian)

A simple constant multiple of $y_1(t)$ won't work.

(Recall, for general soln $y(t) = c_1 y_1(t) + c_2 y_2(t)$ need a fundam. set y_1, y_2)

Idea: Try $y_2(t) = \underbrace{w(t)}\, y_1(t)$ some function that would "work"

For this to be a soln to ODE, it has to satisfy $\circledast$

Compute derivatives of y_2, plug into $*$:

$$\begin{cases} y_2(t) = w(t)e^{-t} \\ y_2'(t) = w'(t)e^{-t} - w(t)e^{-t} \\ y_2''(t) = w''(t)e^{-t} - 2w'(t)e^{-t} + w(t)e^{-t} \end{cases}$$

plug into $*$, cancel e^{-t} from both sides to get

$$\overbrace{\left[w''(t) - 2\cancel{w'(t)} + \cancel{w(t)}\right]}^{y''} + 2\overbrace{\left(\cancel{w'(t)} - \cancel{w(t)}\right)}^{y'} + \overbrace{\cancel{w(t)}}^{y} = 0$$

Note: terms cancel leaving $w''(t) = 0$! (much simpler)

$$\Rightarrow \quad w'(t) = k_1 \qquad \Rightarrow \qquad w(t) = k_1 t + k_2$$

So $\tilde{y}_2(t) = [k_1 t + k_2]e^{-t}$ is another soln.

but since we know $y_1(t) = e^{-t}$ is already a soln, by superposition we could chose $y_2(t) = \dfrac{\tilde{y}_2 - k_2 y_1}{k_1} = te^{-t}$ as our second soln, Conclusion: If roots repeated $(r=-1,-1)$, the two solns are

$$y_1(t) = e^{-t} \qquad y_2(t) = te^{-t}$$

→ can use I.C's to show that $y(t) = c_1 e^{-t} + c_2 te^{-t}$

$$\begin{cases} 1 = y(0) = c_1 \\ 3 = y'(0) = -c_1 + c_2 \end{cases} \rightarrow c_1 = 1, c_2 = 4$$

$$y(t) = e^{-t} + 4te^{-t}$$

This idea works more generally. For

$$ay'' + by' + cy = 0$$

when $b^2 - 4ac = 0$ roots are $r = -\frac{b}{2a}$ (repeated)

then $y_1(t) = e^{-\frac{b}{2a}t}$

by same process we'd find that a 2nd soln is

$$y_2(t) = t\,e^{-\frac{b}{2a}t}$$

Do these form a fundamental set??

check Wronskian:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-\frac{b}{2a}} & te^{-\frac{b}{2a}} \\ -\frac{b}{2a}e^{-\frac{b}{2a}} & (1-\frac{b}{2a}t)e^{-\frac{b}{2a}} \end{vmatrix} = \dots = e^{-\frac{b}{2a}t} \neq 0$$

↑ some algebra

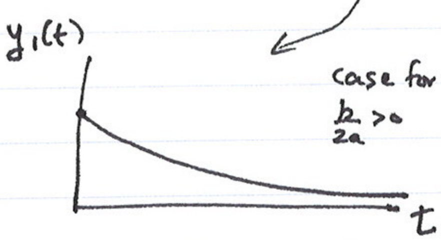
See p168 B+D, 9th ed. Boyce diPrima

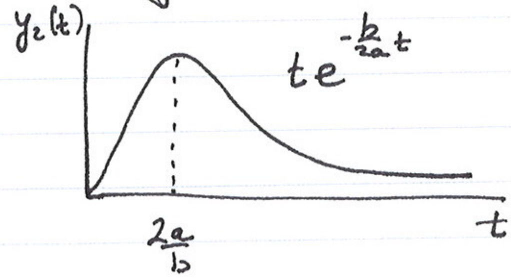
$W \neq 0$ so yes!

⇒ gen'l soln is

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t}$$

What do these solns look like?





Nonhomogeneous Eqns

See B+D pp 174-181 9th ed.

First the general idea:

(could be nonconstant coeffs)

Consider $y'' + p(t)y' + q(t)y = \boxed{g(t)}$ ← "forcing function" or applied input
* by analogy to force applied to spring-mass system

Eqn is called <u>nonhomogeneous</u> whenever $g(t) \neq 0$

The corresponding homog. problem is

$$y'' + p(t)y' + q(t) = 0$$

Then Thm 3.5.2 p175 states that the general soln to nonhom. prob. is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y_p(t)$$

where these are a fundam. set of solns to the homog. problem

and this is any <u>particular soln</u> to the nonhomog problem.

We know how to find this part (see all preceding lectures) in the case of constant coeffs. ($p(t)$, $q(t)$ constant)

How do we find this?

Use METHOD OF UNDETERMINED COEFFICIENTS

Finding a Particular Soln using UNDETR' COEF.

Example ① Solve $y'' - 2y' - 3y = 3e^t$ ← $g(t)$

Note: Corresponding homog. problem $y'' - 2y' - 3y = 0$ has solns e^{3t}, e^{-t}

To find a particular soln try

$Y_p(t) = Ae^t$ ← "looks like" forcing term, but we have to find A so this works

then $Y_p'(t) = Y_p''(t) = Ae^t$

plug into nonhomog. eqn: $\underset{y''}{Ae^t} - 2(\underset{y'}{Ae^t}) - 3(\underset{y}{Ae^t}) = 3e^t$

cancel e^t, simplify: $A(1-2-3) = 3 \Rightarrow A = \frac{-3}{4}$

So general soln is: $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y_p(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4}e^t$

Example 2 : Suppose we have a forcing function that now looks slightly different, i.e. $g(t) = 2e^{-t}$:

previously we had $3e^{+t}$ for $g(t)$

We want to find a particular soln to

$$y'' - 2y' - 3y = 2e^{-t}$$

Suppose we try $Y_p(t) = Ae^{-t}$ ($\leftarrow$ similar to form of $g(t)$)

We encounter a contradiction! (HW3)

This stems from the fact that e^{-t} is already a soln to homog. problem.

In this case, we revise our guess, use $Y_p(t) = Ate^{-t}$

It turns out that this will work. (HW 3)