Sept 22 - Linear 2nd order ODEs, cont'd

Last time: (*) $ay'' + by' + cy = 0$ — a, b, c constants coefficients

homogeneous i.e. no $f(t)$

applications include:

spring-mass system

a = mass m
b = drag coef. c
c = spring const. k

LRC circuit

a = inductance L
b = resistance R
$c = \frac{1}{\text{capacitance}}$

also noted that

$$y(t) = e^{rt} \quad r \text{ constant} \quad \text{will be a soln to } (*)$$

r satisfies $ar^2 + br + c = 0$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cases:

$b^2 - 4ac > 0$ two real distinct roots $e^{r_1 t}, e^{r_2 t}$

$b^2 - 4ac = 0$ $r = r_{1,2} = \frac{-b}{2a}$ only one root (roots same) e^{rt}

$b^2 - 4ac < 0$ two complex conjugate roots $r_{1,2} = \sigma \pm i\omega$

$(\sigma \pm i\omega)t$

$e^{(\sigma \pm i\omega)t}$

$$\sigma = \frac{-b}{2a}, \quad \omega = \frac{\sqrt{|b^2 - 4ac|}}{2a}$$

$$e^{\sigma t} \sin \omega t, \quad e^{\sigma t} \cos \omega t$$

General Results about 2nd order linear ODEs.

$$* \quad y'' + p(t)y' + q(t)y = \underbrace{g(t)}$$

p, q, "nonconstant coeffs"

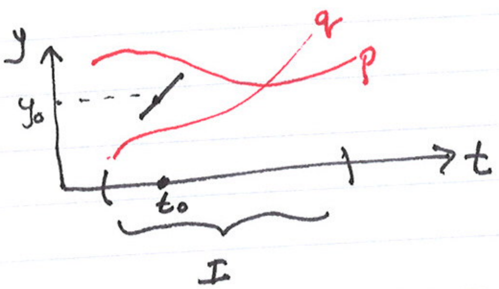
nonhomogeneous ODE.

Thm 3.2.1 p 146 (B+D.)

Consider $*$ with initial conds $y(t_0) = y_0$, $y'(t_0) = y_0'$ } known i.e. given

and $p(t)$, $q(t)$, $g(t)$ "well behaved"

↑ continuous fn on interval I



Then there exists a unique soln to ODE + I.C. defined on whole interval I.

- there is a soln.
- only one soln.
- defined on all I

} contrast with nonlin ODEs that can fail HW 2 → last Q.

Now consider Homogeneous case:

$$y'' + p(t)y' + q(t)y = 0$$

Principle of linear superpos.: Suppose $y_1(t)$, $y_2(t)$ are two solutions to this ODE. Then

$$y(t) = \underbrace{c_1 y_1(t) + c_2 y_2(t)}$$ is also a soln. (HW.2)

"linear comb."
"lin superpos"

c_1, c_2 are constants.

$$\underbrace{y'' - y = 0}_{\text{ODE}} \qquad \underbrace{y(0) = 2 \quad y'(0) = +1}_{\text{IC}}$$

Example:

Homog.

const coef. $a = 1$, $b = 0$, $c = -1$

$p(t) =$

$q(t) =$

$g(t) = 0$

plug in $y(t) = e^{rt}$ $\qquad y''(t) = r^2 e^{rt}$

$$\boxed{r^2 - 1 = 0}$$ char. eqn. , $r = \pm 1$

$y_1(t) = e^t \qquad y_2(t) = e^{-t}$ will be solns.

$\therefore \quad y(t) = c_1 e^t + c_2 e^{-t}$ should also be a soln. to ODE. (the general soln)

Ques: How to find c_1, c_2? Ans: Use I.C.

need $y'(t) = c_1 e^t - c_2 e^{-t}$

$y(0) = 2 \Rightarrow 2 = c_1 e^0 + c_2 e^0 \Rightarrow \begin{cases} 2 = c_1 + c_2 \\ 1 = c_1 - c_2 \end{cases}$

$y'(0) = \qquad 1 = c_1 - c_2$

Solve this

$c_2 = 1/2 \quad c_1 = 3/2$

Remarks:
- could use I.C. at any other time $t = t_0$ not only $t = 0$
- 2 I.C. $\Longleftrightarrow$ unique value of c_1, c_2.
- soln is then [in this example]:

$$\boxed{y(t) = \frac{3}{2} e^t + \frac{1}{2} e^{-t}}$$

Question: Can we always find unique set c_1, c_2

Algebra review:

$$\left.\begin{aligned} 2 &= c_1 + c_2 \\ 1 &= c_1 - c_2 \end{aligned}\right\}$$ lin. algebr. eqns. unique soln.

Suppose we arrive at:

$$\left.\begin{aligned} 2 &= c_1 + c_2 \\ 10 &= 5c_1 + 5c_2 \end{aligned}\right\}$$ no unique soln.

matrix form

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}}_{M} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

$\det M = 0$

$\Rightarrow$ non uniqueness.

$$\boxed{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc}$$

Same kind of idea: p148 in B+D 9th ed.

given homog. 2nd order ODE and I.C. $\left.\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= y_0' \end{aligned}\right\}$ given.

suppose $y_1(t)$, $y_2(t)$ are two solns to ODE.

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

plan: use I.C. to find c_1, c_2

We can do this provided Wronskian $\neq 0$

i.e. $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \neq 0$$

in that case y_1, y_2 form a fundamental set of solutions to ODE