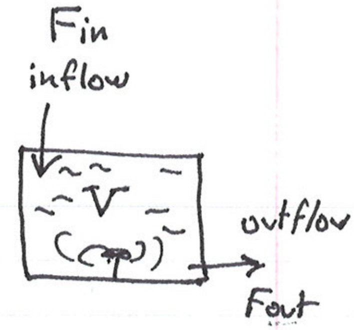
Sept 13 First order linear ODE's cont'd

Example 1: Stirred tank reactor



inflow of stock conc. S of salt

rate inflow F_{in}

" outflow F_{out}

Conc of salt in tank $C(t)$ ← want to understand.

Assume: - vol. well stirred

- $F_{in} = F_{out}$ (vol doesn't change) (but see HW 1 last prob.)

Units: S, $C(t)$ conc. gm / Litre

V vol Litre

t time hr.

$F = F_{in} = F_{out}$ flow rate Litre / hr

* MASS CONSERVATION

i.e. total rate of change of mass (of salt) $= 0$

Rate of change of mass in tank = rate inflow salt − rate outflow salt.

mass of salt in tank $= C(t) \cdot V$ gm.

rate mass coming in $= S \cdot F_{in} = SF$ gm/h

" " leaves $= C(t) \cdot F_{out} = C(t) F$ "

$$\frac{d}{dt}\left[C(t) V\right] = SF - C(t) F$$ (check consistency of eqn units)

Vol is constant ∴

$$\frac{dC}{dt} = \underbrace{\left(\frac{SF}{V}\right)}_{a} - \underbrace{\left(\frac{F}{V}\right)}_{b} C(t)$$

rename:

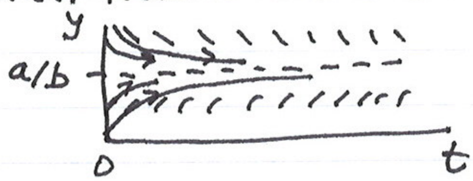
We arrived at eqn of form

ODE: $\frac{dy}{dt} = a - by$ ⊛ want: find $y(t)$

IC: $y(0) = 0$

Class of well studied linear ODES

<u>Remarks</u>: - could study using direc. fields.



- $y \to \frac{a}{b}$ [steady state] as $t \to \infty$

- in case $a=0$, know solns exp. decreasing (example 1 Sept8)

- this eqn is "separable"

$$\frac{dy}{a-by} = dt \quad \ldots \text{ integrate + solve.}$$

- Thought: what if $F_{in} \neq F_{out}$? what to do? HW1
- $F_{in} = F(t)$? ← later.

Technique for solving ⊛ any linear 1st order ODE.

INTEGRATING FACTOR

<u>Example</u>: $\frac{dy}{dt} = a - by$

$$\frac{dy}{dt} + by = a \quad \text{"standard form"}$$

Try to find a function $\mu(t)$ s.t.

$$\underbrace{\mu(t)\left[\frac{dy}{dt} + by\right]}_{} = \mu(t)\, a \quad \text{is easy to integrate}$$

$$\frac{d}{dt}\left[\mu(t)\, y\right] = \mu(t)\, a \quad \text{then just } \int dt \text{ will do it.}$$

$$\mu(t)\frac{dy}{dt} + y\frac{d\mu}{dt}$$

will work if $\mu(t)\, b\cancel{y} = \cancel{y}\frac{d\mu}{dt}$

$$\frac{d\mu}{dt} = b\mu$$

$$\frac{d\mu}{\mu} = b\,dt$$

$$\ln \mu = \int b\,dt$$

in general: $\boxed{\mu(t) = \exp\left[\int b\,dt\right]}$

if $b = \text{constant}$
$\mu(t) = \exp[bt]$
$= e^{bt}$

in the example:

$$\underbrace{\mu(t)\left[\frac{dy}{dt} + by\right]} = \mu(t)\,a$$

$$\frac{d}{dt}\left[\mu(t)\,y\right] = \mu(t)\,a$$

$$\frac{d}{dt}\left[e^{bt}y\right] = e^{bt}a$$

integrate:

const of integr.

$$e^{bt}y = \int e^{bt}a\,dt + K$$

$$= \frac{a}{b}e^{bt} + K$$

isolate y:

$$y(t) = e^{-bt}\left[\quad " \quad " \quad\right]$$

$$y(t) = \frac{a}{b} + Ke^{-bt}$$

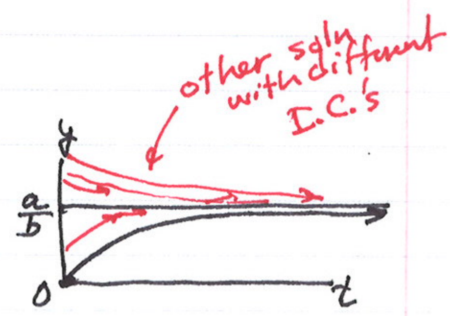
This is the general soln to ODE.
Now use I.C. $\underset{\substack{\uparrow \\ t=0}}{y(0)} = \underset{\substack{\uparrow \\ y=0}}{0}$ Find K.

$$0 = \frac{a}{b} + Ke^{-b\cdot 0} = \frac{a}{b} + K$$

$$K = -\frac{a}{b} \implies y(t) = \frac{a}{b} - \frac{a}{b}e^{-bt}$$

$b > 0$

$$\boxed{y(t) = \frac{a}{b}\left(1 - e^{-bt}\right)}$$



What does it tell us about tank problem?

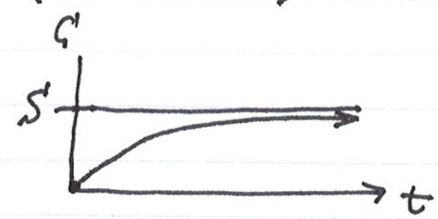
$$a = \frac{S'F}{V} \qquad b = \frac{F}{V} \qquad \Rightarrow \qquad \frac{a}{b} = S'$$

$c(0) = 0$ (no salt in tank initially)

$$c(t) \Leftrightarrow y(t)$$

b

get $c(t) = S'\left(1 - e^{-\frac{F}{V}t}\right)$



the approach to steady state $c(t) = S'$ is "on a timescale" $\frac{V}{F}$

$$c(t) = S'\left(1 - e^{\frac{t}{\tau}}\right)$$

<u>Example 2</u> p39 #15

must have $t > 0$

$$t\frac{dy}{dt} + 2y = t^2 - t + 1 \qquad y(1) = \tfrac{1}{2}$$

put in st. form:

$$\frac{dy}{dt} + \underset{\underset{b(t)}{\uparrow}}{\left(\frac{2}{t}\right)} y = \underbrace{\left(\frac{t^2 - t + 1}{t}\right)}_{\underset{a(t)}{\uparrow}}$$

on line

$$\mu(t) = \exp\left[\int b(t)\,dt\right] = \exp\left[\int \frac{2}{t}\,dt\right] = \dots t^2$$

$$\frac{d}{dt}\left[t^2 y\right] = t^2\left(\quad " \quad \right)$$

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{K}{t^2} \qquad K = \frac{1}{12}$$