Examples for Review

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Example: 1st order ODE, integr. factor

$$\frac{dc}{dt} = I - rc \qquad I, r > 0, \text{ constant} \qquad c(0) = 0$$

$$\frac{dc}{dt} + rc = I \qquad \text{integr. factor } \mu(t) = \exp\left[\int r\,dt\right] = e^{rt}$$

$$\frac{d[e^{rt}c]}{dt} = e^{rt} I$$

arbitr. const.

$$e^{rt} c = I \int e^{rt} dt + K$$

$$e^{rt} c(t) = \frac{I}{r} e^{rt} + K$$

$$c(t) = \frac{I}{r} + K e^{-rt}$$

Initial cond's: $c(0) = 0 \Rightarrow 0 = \frac{I}{r} + K \qquad K = -\frac{I}{r}$

$$c(t) = \frac{I}{r} - \frac{I}{r} e^{-rt} = \frac{I}{r}(1 - e^{-rt})$$

Example: Nonhomog. 2nd order ODE: Solve $y'' - 4y' - 5y = 10t \quad (*)$

- Hom. probl: $y'' - 4y' - 5y = 0$

  char eq: $r^2 - 4r - 5 = 0 \qquad (r-5)(r+1) = 0$

  $r = 5, -1 \rightarrow e^{5t}, e^{-t}$

  soln to Hom: $y_h = c_1 e^{5t} + c_2 e^{-t}$

- Particular soln $\rightarrow Y_p(t) = At + B$

  its derivatives $\begin{cases} Y_p'(t) = A \\ Y_p''(t) = 0 \end{cases}$ $\rightarrow$ plug into $*$

  $$0 - 4(A) - 5(At + B) = 10t$$

  sort terms

  $$t(-5A) + (-4A - 5B) = 10t$$

  $-5A = 10 \qquad A = -2$

  $-4A - 5B = 0 \qquad B = \frac{8}{5}$

- Gen'l soln

  $$y(t) = c_1 e^{5t} + c_2 e^{-t} - 2t + \frac{8}{5}$$

- If initial conditions are given, here is where we use them to find c_1, c_2. E.g. if $y(0) = 0$, $y'(0) = 0$ then

  $$\left.\begin{array}{l} 0 = y(0) = c_1 + c_2 + 8/5 \\ 0 = y'(0) = 5c_1 - c_2 + 2 \end{array}\right\} \Longrightarrow \quad c_1 = \frac{1}{15} \quad c_2 = -\frac{5}{3}$$

<u>Example</u> Nonhomog. 2nd order ODE

Solve: $y'' - 4y' - 5y = e^{-t}$ $y(0)=0$ $y'(0)=0$

hom. equ: $y'' - 4y' - 5y = 0$

char eqn: $r^2 - 4r - 5 = 0$ $r = +5, -1 \Rightarrow e^{-t}, e^{5t}$

Guess for particular soln $\rightarrow$ $Y_p = A t e^{-t}$ need factor of t to avoid duplicating this

$Y_p' = A(e^{-t} - te^{-t}) = A(e^{-t}(1-t))$

$Y_p'' = A(-e^{-t} - e^{-t} + te^{-t}) = Ae^{-t}(-2+t)$

plug into nonhom. ODE $A\cancel{e^{-t}}(-2+t) - 4A\cancel{e^{-t}}(1-t) - 5At\cancel{e^{-t}} = \cancel{e^{-t}}$

$A(-2-4) = 1 \longrightarrow -6A = 1$ $A = -\frac{1}{6}$

$At(1+4-5) = 0$ $0 = 0$ ✓

The particular solution we want is $Y_p(t) = -\frac{1}{6} te^{-t}$

$y(t) = c_1 e^{-t} + c_2 e^{5t} - \frac{1}{6} te^{-t}$

Initial conds: find c_1, c_2

$$\begin{cases} 0 = y(0) = c_1 + c_2 + 0 & \Rightarrow c_1 = -c_2 \\ 0 = y'(0) = c_1(-1) + c_2(5) + \left(-\frac{1}{6}\right) \\ 0 = c_2 + 5c_2 - \frac{1}{6} \\ \frac{1}{6} = 6c_2 \qquad c_2 = \frac{1}{36} \qquad c_1 = -\frac{1}{36} \end{cases}$$

<u>Solution:</u> $y(t) = -\frac{1}{36} e^{-t} + \frac{1}{36} e^{5t} - \frac{1}{6} te^{-t}$