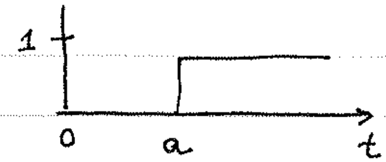
Laplace Transform, Cont'd

Step functions:

We first introduce the unit step function



$$u_a(t) = \begin{cases} 0 & t \le a \\ 1 & t > a \end{cases}$$

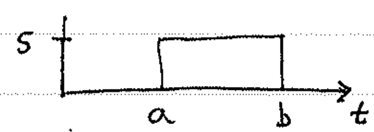
Alternate notations often used for same function:

$u(t-a)$

Heaviside $(t-a)$ — all mean same thing

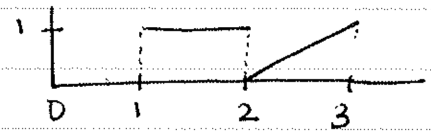
$H(t-a)$

This function is useful for describing a variety of piecewise continuous functions.

Examples: ① 

$\leftarrow y = 5\left[H(t-a) - H(t-b)\right]$

$= 5\left(u_a(t) - u_b(t)\right)$

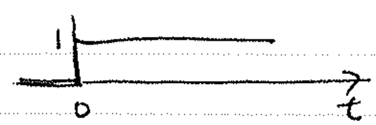
② 

$y = u_1(t) - u_2(t) + (t-2)(u_2 - u_3)$

also written as $y = H(t-1) - H(t-2) + (t-2)\left(H(t-2) - H(t-3)\right)$

Remark: the Heaviside function (named after Oliver Heaviside) is

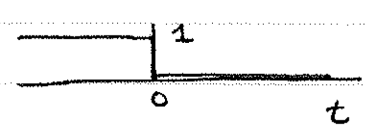
$$H(t) \equiv \text{Heaviside}(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0 \end{cases}$$



In engineering such functions are useful since they represent the (abrupt) turning "ON" of a switch or discontinuous input.
We can also combine these with other functions as we will see.

And here is $1 - H(t)$ $\longrightarrow$

(also written $1 - u_0(t)$)



Q: Find the Laplace transform of the unit step function $u_a(t) = H(t-a)$

A: $\mathcal{L}\{u_a(t)\} = \int_0^\infty u_a(t)\, e^{-st}\, dt = \int_0^a \underbrace{u_a(t)}_{\substack{= \\ 0 \\ \text{on this interval}}} e^{-st}\, dt + \int_a^\infty \underbrace{u_a(t)}_{\substack{= \\ 1 \text{ on this interv.}}} e^{-st}\, dt$

$= \int_a^\infty 1 \cdot e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right|_a^\infty = \frac{-1}{s}\Big[\underbrace{\lim_{t\to\infty} e^{-st}}_{\substack{= \\ 0 \text{ for } s>0}} - e^{-as}\Big]$

$= -\frac{1}{s}\left[-e^{-as}\right] = \frac{e^{-as}}{s}$ for $s>0$

Conclude: $\mathcal{L}\{u_a(t)\} \equiv \mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s}$ ⇐ $\boxed{= \frac{1}{s}\, e^{-as}}$

just different notations for step fn

Note: this is same as $e^{-as}\mathcal{L}\{1\}$

Q: Find the Laplace transform of the function in Example 1

A: $\mathcal{L}\{f(t)\} = \mathcal{L}\{5[H(t-a) - H(t-b)]\}$ ← using the alternate notation, just to get used to both

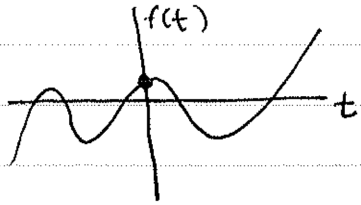
$= 5\left(\mathcal{L}\{H(t-a)\} - \mathcal{L}\{H(t-b)\}\right)$

$= 5\left(\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right) = \frac{5}{s}\left(e^{-as} - e^{-bs}\right)$

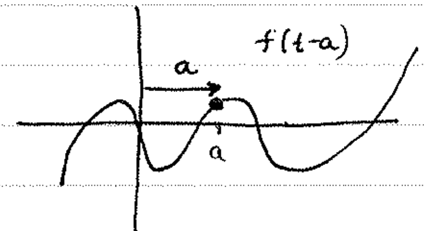
Shifts and steps

Given some function $f(t)$. Suppose its graph is:

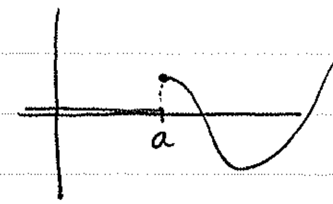
Sketch $f(t-a)$ and $u_a(t)\cdot f(t-a)$



Original graph



This is a horizontal shift by $+a$



This is a shifted copy turned "on" at a.

Laplace Transform Shift Theorem ①:

Shift Thm: ① $\boxed{\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}\,\mathcal{L}\{f(t)\} = e^{-as}F(s)}$ for $a>0$

Verify:

$$= \int_0^\infty f(t-a)\,u_a(t)\,e^{-st}\,dt = \underbrace{\int_0^a}_{0} + \int_a^\infty e^{-st} f(t-a)\,\underbrace{u_a(t)}_{1}\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt \qquad \text{now let } \tau = t-a$$

$$\leftarrow \begin{cases} d\tau = dt \\ -st = -s(\tau+a) \end{cases}$$

$$= \int_0^\infty e^{-s(\tau+a)} f(\tau)\,d\tau$$

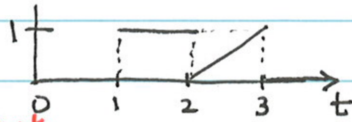
$$= e^{-sa}\int_0^\infty \underbrace{e^{-s\tau} f(\tau)\,d\tau}_{} = e^{-as}\,\mathcal{L}\{f(t)\}$$

dummy var. $\qquad = e^{-as}F(s)$

Remark: Can also write this as

$$\boxed{\mathcal{L}\{H(t-a)\,f(t-a)\} = e^{-as}F(s)}$$

Example: Find the Laplace Tr of the fn on previous page / i.e. in Example ②



$$f(t) = u_1 - u_2 + (t-2)\left[u_2 - u_3\right]$$

$$= u_1 - u_2 + (t-2)u_2 - \boxed{(t-2)u_3} \;\; *$$

$$= u_1 - u_2 + (t-2)u_2 - \boxed{(t-3)u_3 - u_3}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_1\} - \mathcal{L}\{u_2\} + \mathcal{L}\{(t-2)u_2\} - \boxed{\mathcal{L}\{(t-3)u_3\} - \mathcal{L}\{u_3\}}$$

$$= \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} + e^{-2s}\mathcal{L}\{t\} - e^{-3s}\mathcal{L}\{t\} - \frac{e^{-3s}}{s}$$

$$= \frac{1}{s}\left(e^{-s} - e^{-2s} - e^{-3s}\right) + \frac{1}{s^2}\left(e^{-2s} - e^{-3s}\right)$$

* **Remark:** Notice how we handle the term $\boxed{-(t-2)u_3(t)}$ by rewriting it as $\boxed{-(t-3)u_3(t) - u_3(t)}$ so we can use the Shift thm.

Q: Find the Laplace transform:

$$f(t) = t e^{-3t} H(t-1)$$

Recall $\mathcal{L}\{e^{at} f(t)\} = F(s-a) = F(s)\big|_{s \to s-a}$

A: $\mathcal{L}\{f(t)\} = \mathcal{L}\{t e^{-3t} H(t-1)\} = \mathcal{L}\{t\, H(t-1)\}\Big|_{s \to s+3}$

(rewrite so we can use the last result) $\mathcal{L}\{H(t-a) f(t-a)\} = e^{-as} F(s)$

$$= \mathcal{L}\{(t-1) H(t-1) + H(t-1)\}\Big|_{s \to s+3}$$

$$= \left(\mathcal{L}\{t\} e^{-s} + \mathcal{L}\{1\} e^{-s}\right)\Big|_{s \to s+3}$$

$$= \left[e^{-s} \cdot \frac{1}{s^2} + e^{-s} \cdot \frac{1}{s}\right]\Bigg|_{s \to s+3}$$

$$= \frac{e^{-(s+3)}}{(s+3)^2} + \frac{e^{-(s+3)}}{(s+3)}$$ (Typo corrected in signs)

Q Find inverse Laplace transform. of $F(s) = \dfrac{s}{s^2+6s+11}$

Soln: Complete the square

$$F(s) = \frac{s}{s^2+6s+11} = \frac{s}{(s^2+6s+9)+2} = \frac{s}{(s+3)^2+2} = \frac{s+3-3}{(s+3)^2+2}$$

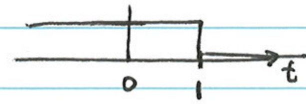
$$= \frac{(s+3)}{(s+3)^2+2} - \frac{3}{(s+3)^2+2} = \frac{s}{s^2+2}\Bigg|_{s \to s+3} - \frac{3}{\sqrt{2}} \frac{\sqrt{2}}{s^2+2}\Bigg|_{s \to s+3}$$

$$\mathcal{L}^{-1}\{F(s)\} = e^{-3t} \cos(\sqrt{2}\, t) - \frac{3}{\sqrt{2}} e^{-3t} \sin(\sqrt{2}\, t)$$

Solve the following IVPs using the Laplace Transform.

$y(0)=1,\ y'(0)=0$

$$y'' + 2y' - 3y = H(1-t) \qquad = 1 - H(t-1)$$



↑note: this turns off at t=1

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{H(1-t)\} = \mathcal{L}\{1\} - \mathcal{L}\{H(t-1)\}$$

$$\Big[s^2F(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0}\Big] + 2\Big[sF(s) - \underbrace{y(0)}_{1}\Big] - 3F(s) = \frac{1-e^{-s}}{s}$$

$$F(s)\left(s^2+2s-3\right) = \frac{1-e^{-s}}{s} + s + 2$$

$$F(s) = \frac{1}{(s^2+2s-3)}\left[\frac{1-e^{-s}}{s} + s + 2\right]$$

$$= \frac{1}{(s+3)(s-1)}\left[\frac{1-e^{-s}+s^2+2s}{s}\right]$$

$$= \frac{1+s^2+2s}{s(s+3)(s-1)} - \frac{1}{s(s+3)(s-1)}e^{-s}$$

$$\frac{A}{s} + \frac{B}{(s+3)} + \frac{C}{s-1} = \frac{1+s^2+2s}{s(s+3)(s-1)}$$

partial fractions (of each part)

$$A(s-1)(s+3) + B(s)(s-1) + Cs(s+3) = 1+s^2+2s$$

$s\to 1$: $\quad C\cdot 4 = 4 \qquad C=1$

$s\to -3$ $\quad B(-3)(-4) = 1+9-6 = 4 \qquad B = \frac{1}{3}$

$s\to 0$ $\quad A(-1)(3) = 1 \qquad A = -1/3$

[same things] $= 1$

$C\cdot 4 = 1 \qquad C = 1/4$

$B(12) = 1 \qquad B = 1/12$

$A(-3) = 1 \qquad A = -1/3$

$$F(s) = \frac{-1/3}{s} + \frac{1/3}{s+3} + \frac{1}{s-1} \longrightarrow - + - \longrightarrow -e^{-s}\left(\frac{-1/3}{s} + \frac{1/12}{s+3} + \frac{1/4}{s-1}\right)$$

$$f(t) = -\frac{1}{3} + \frac{1}{3}e^{-3t} + e^{t} - \left(-\frac{1}{3} + \frac{1}{12}e^{-3t} + \frac{1}{4}e^{t}\right)_{t\to t-1}$$

$$= -\frac{1}{3} + \frac{1}{3}e^{-3t} + e^{t} - H(t-1)\left(-\frac{1}{3} + \frac{1}{12}e^{-3(t-1)} + \frac{1}{4}e^{(t-1)}\right)$$

Summary of Shift Theorems

suppose $F(s) = \mathcal{L}\{f(t)\}$

① $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

② $e^{-as} \mathcal{L}\{f(t)\} = \mathcal{L}\{H(t-a) f(t-a)\}$ $\quad a \geq 0$

③ (Restatement of ②)

$$e^{-as} \mathcal{L}\{\underbrace{f(t+a)}_{\text{call it } g(t)}\} = \mathcal{L}\{H(t-a) f(t)\}$$

$$e^{-as} \mathcal{L}\{g(t)\} = \mathcal{L}\{H(t-a) f(t)\} \quad \text{where} \quad g(t) = f(t+a)$$

$$g(t-a) = f(t)$$

②' $\mathcal{L}^{-1}(e^{-as} F(s)) = H(t-a) f(t-a)$

①' $\mathcal{L}^{-1}(F(s-a)) = e^{at} f(t)$