Oct 20, 2010
Math 265

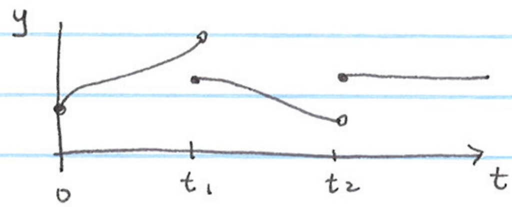
Q: For what functions will Laplace transform exist?

Given: • $f(t)$ that is piecewise continuous* on $0 \le t \le A$ for all $A > 0$

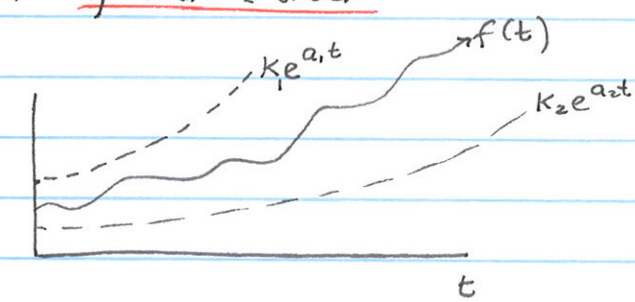
• $|f(t)| \le K e^{at}$ when $t \geq M$ for K, a, M real, $K, M > 0$

Then $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$

This type of function is said to be "of exponential order"
we also refer to it as "acceptable".



piecewise continuous



Laplace Tr will exist for $s > a_1$

Examples of acceptable functions:

- power functions
- polynomials
- $\sin(at)$, $\cos(at)$
- e^{at}
- sums and products of these.

Examples of unacceptable functions:

$f(t) = e^{t^2}$ (grows faster than Ke^{at} for any K and a)

$f(t) = \frac{K}{10 - t}$ ("blows up" at $t = 10$ i.e. not piecewise continuous on any interval containing $t = 10$)

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* A function is piecewise continuous on $[a, b]$ if f is continuous at every point in $[a, b]$ except possibly at a finite number of points at which it has a jump discontinuity.

example: $f(t) = \frac{1}{t}$ is not piecewise cont' on any interval containing $t = 0$

Computing some Laplace Transforms:

① $f(t) = K =$ constant

$$\mathcal{L}\{f(t)\} = K\int_0^\infty e^{-st}\,dt = \frac{K}{-s}\,e^{-st}\Big|_0^\infty = \frac{K}{-s}\left(\lim_{t\to\infty} e^{-st} - e^0\right)$$

$$= \frac{K}{-s}(0-1) = \frac{K}{s} \quad \text{for } s>0$$

$\lim_{t\to\infty} e^{-st} = 0$ for $\underline{s>0}$

Remark: we already saw this earlier in the improper integral ③. Note that constant K can "come out"

$$\mathcal{L}\{K\} = K\,\mathcal{L}\{1\} = K\cdot\frac{1}{s} = \frac{K}{s} \qquad \text{(linearity)}$$

② $f(t) = e^{at}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{at}e^{-st}\,ds = \int_0^\infty e^{(a-s)t}\,dt = \frac{1}{a-s}\,e^{(a-s)t}\Big|_0^\infty$$

$$= \frac{1}{a-s}\left[\lim_{t\to\infty} e^{(a-s)t} - e^0\right]$$

$$= \frac{1}{a-s}\left[\,0 \quad -1\,\right] \quad \text{for } s>a$$

$$= \frac{1}{s-a} \qquad s>a$$

③ $f(t) = Kt$

$$\mathcal{L}\{f(t)\} = \int_0^\infty Kt\,e^{-st}\,dt = \ldots = \frac{K}{s^2}, \quad s>0 \qquad \text{(HW5)}$$

use integr. by parts and $\lim_{t\to\infty} te^{-st} = 0$ for $s>0$

$f(t)$	$F(s)$	
1	$\frac{1}{s}$	✓
δ	1	
$\delta^{(k)}$	s^k	
t	$\frac{1}{s^2}$	← (HW5)
$\frac{t^k}{k!}$, $k \geq 0$	$\frac{1}{s^{k+1}}$	
e^{at}	$\frac{1}{s-a}$	✓
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	← (HW5)
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	
$\cos(\omega t + \phi)$	$\frac{s \cos \phi - \omega \sin \phi}{s^2+\omega^2}$	
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	

Laplace Tr. of derivatives

Let $F(s) = \mathcal{L}\{f(t)\}$ be the Laplace transform of $f(t)$.

Find $\mathcal{L}\{\frac{df}{dt}\}$. (also written $\mathcal{L}\{f'(t)\}$)

Soln: $$\mathcal{L}\{f'(t)\} = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv}$$

← integrate by parts

$u = e^{-st}$ $du = -se^{-st}dt$

$dv = f'(t)dt$ $v = f(t)$

$$= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty (-s) f(t) e^{-st} dt$$

$$= \lim_{t\to\infty} f(t)e^{-st} - f(0)e^0 + s\int_0^\infty f(t)\,e^{-st}dt$$

assumes f is "of exponential order"

$$= 0 - f(0) + s\,F(s)$$

so $$\boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0)}$$

carefully note these terms

similarly $$\boxed{\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}$$

(typo corrected)

In general

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots f^{(n-1)}(0)$$

nth derivative of f

n-1 "initial conds"

Use Laplace transform to solve

$$y'' + 4y = 4t \qquad y(0) = 1\ , \quad y'(0) = 5$$

$$\mathcal{L}\{y''\} + 4\,\mathcal{L}\{y\} = 4\mathcal{L}\{t\}$$

$$\left(s^2 F(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{5}\right) + 4F(s) = \frac{4}{s^2}$$

$$\left(s^2 F(s) - s - 5\right) + 4F(s) = \frac{4}{s^2}$$

$$F(s)\,(4+s^2) = \frac{4}{s^2} + s + 5$$

$$F(s) = \frac{1}{4+s^2}\left(\frac{4 + s^3 + 5s^2}{s^2}\right)$$

every term

$$= \frac{s}{4+s^2} + \frac{5}{4+s^2} + \frac{4}{s^2(4+s^2)}$$

← break up into partial fraction form

$$\boxed{\begin{aligned} \frac{4}{s^2(4+s^2)} &= \frac{A}{s^2} + \frac{B}{4+s^2} = \frac{A(4+s^2) + Bs^2}{s^2\,(4+s^2)} \\ 4 &= A(4+s^2) + Bs^2 \implies \left.\begin{aligned} 4 &= 4A \\ 0 &= A+B \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 1 \\ B &= -1 \end{aligned} \end{aligned}}$$

$$F(s) = \frac{s}{4+s^2} + \frac{5}{4+s^2} + \frac{1}{s^2} - \frac{1}{4+s^2}$$

$$= \frac{s}{4+s^2} + \frac{4}{4+s^2} + \frac{1}{s^2}$$

$$\downarrow$$

$$y(t) = \mathcal{L}^{-1}\{F(s)\} = \cos 2t + 2\sin 2t + t$$

Some Important Facts about the Laplace Transform

(HW5)

basic transforms

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

<u>Derivatives</u>: If $F(s) = \mathcal{L}\{f(t)\}$

then
$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$
$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

<u>Shifts</u>

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

<u>Derivatives of the transform</u>
$$\frac{dF(s)}{ds} = -\mathcal{L}\{t f(t)\}$$

<u>Linearity</u>

- $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$
- $\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}$

(for a, b constants, and f, g acceptable fns)

Solve $y'' + y' - 2y = 4e^t + 1 \qquad y(0) = 1 \,,\ y'(0) = 0$

$$\mathcal{L}\{y'' + y' - 2y\} = \mathcal{L}\{4e^t + 1\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - \mathcal{L}\{2y\} = 4\mathcal{L}\{e^t\} + \mathcal{L}\{1\}$$

$$\left[s^2 F(s) - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0}\right] + \left[sF(s) - \underbrace{y(0)}_{=1}\right] - 2F(s) = \frac{4}{s-1} + \frac{1}{s}$$

$$s^2F(s) - s + sF(s) - 1 - 2F(s) = \frac{4}{s-1} + \frac{1}{s}$$

$$F(s)\left(s^2 + s - 2\right) - (s+1) = \frac{4}{s-1} + \frac{1}{s}$$

$$F(s) = \left(\frac{1}{s^2+s-2}\right)\left[\frac{4}{s-1} + \frac{1}{s} + s+1\right]$$

Now we want to find the inverse transform, i.e. get $y(t)$
But to do so, need to write $F(s)$ in a form where we can easily use look-up table of functions and their Laplace transform

Steps: (1) Factor denominator fully :

$$F(s) = \frac{1}{(s+2)(s-1)}\left[\frac{4}{s-1} + \frac{1}{s} + s+1\right]$$

(2) Rewrite this in the partial fraction form

$$F(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)}$$

(3) Find A, B, C, D (constants) $\leftarrow$ (HW5)

(4) Look up the functions in table.