Nov 3, 2010
and
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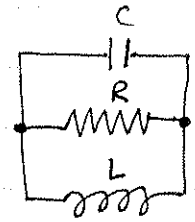
Math 265

Systems of First order Linear Ordinary Differential Equs

Sometimes we are interested in tracking two (or more) variables that affect each other. For example, on p 356 of Boyce + Diprima:

Examples:
① Electric circuit with C, R, L in parallel



$V(t)$ = voltage drop across capacitor
$I(t)$ = current through inductor

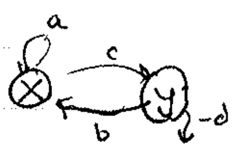
Then (see probl. 19 Sec 7.1) the following two ODES characterize this system:

$$\begin{cases} \dfrac{dI}{dt} = \dfrac{V}{L} \\[2mm] \dfrac{dV}{dt} = -\dfrac{I}{C} - \dfrac{V}{RC} \end{cases}$$

Note that $I(t)$ and $V(t)$ are interdependent or <u>coupled</u>.

When L, R, C are <u>constants</u> this is a linear system of 1st order ODES.

Example ②:



$x(t)$, $y(t)$ are two kinds of cells that can interconvert (e.g. mature vs dormant) and die or reproduce; e.g. x = active cell popul. y = spore popul.

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\[2mm] \dfrac{dy}{dt} = cx - dy \end{cases}$$

might be a system to describe this.
(note x, y are <u>coupled</u>)
a, b, c, d constants (units of t^{-1})
a = growth rate, d = death rate, b, c = conversion rates

We now investigate how to solve such ODE systems.

We first show that there is a lot in common between a 1st order linear <u>system</u> and a 2nd order linear ODE.

Solving a linear system of ODES.

$$\begin{cases} \textcircled{1} \quad \dfrac{dx}{dt} = a_{11}x + a_{12}y & x(0) = x_0 \\[2ex] \textcircled{2} \quad \dfrac{dy}{dt} = a_{21}x + a_{22}y & y(0) = y_0 \end{cases}$$

— Reduction to 2nd order eqn:

eliminate $x(t)$ (or $y(t)$):

$\frac{d}{dt}\textcircled{2}$:

$$\frac{d^2y}{dt^2} = a_{21}\frac{dx}{dt} + a_{22}\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = a_{21}(a_{11}x + a_{12}y) + a_{22}\frac{dy}{dt}$$

from ② $x = \left(\frac{1}{a_{21}}\frac{dy}{dt} - \frac{a_{22}}{a_{21}}y\right)$ $*$

(sub in here)

$$\frac{d^2y}{dt^2} = a_{21}\left[\left(\frac{a_{11}}{a_{21}}\frac{dy}{dt} - \frac{a_{11}a_{22}}{a_{21}}y\right) + a_{12}y\right] + a_{22}\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = (a_{22} + a_{11})\frac{dy}{dt} + (-a_{11}a_{22} + a_{12}a_{21})y$$

$$\boxed{y'' - (a_{11} + a_{22})y' + (a_{11}a_{22} - a_{12}a_{21})y = 0}$$

① $\frac{dx}{dt} = a_{11}x + a_{12}y$

② $\frac{dy}{dt} = a_{21}x + a_{22}y$

can be reduced (by elim. one variable) to:

$$\boxed{\frac{d^2y}{dt} - \underbrace{(a_{11}+a_{22})}_{\beta}\frac{dy}{dt} + \underbrace{(a_{11}a_{22}-a_{12}a_{21})}_{\gamma}y = 0}$$

let $\beta = a_{11}+a_{22}$

$\gamma = a_{11}a_{22} - a_{12}a_{21}$

$$\frac{d^2y}{dt^2} - \beta\frac{dy}{dt} + \gamma y = 0$$

let $y(t) = \boxed{Y_0 e^{rt}}$

(we expect such solns by our experience with 2nd order ODEs)

char eqn $r^2 - \beta r + \gamma = 0$ ← Two possible roots:

note sign. $r_{1,2} = \frac{+\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$ → $\cancel{Y_1}e^{r_1 t}$, $\cancel{Y_2}e^{r_2 t}$

Expect several cases:

$\beta^2 - 4\gamma > 0$ → exponential solns

→ $\gamma > 0$, $\beta > 0$ grow

→ $\gamma > 0$ $\beta < 0$ decay

→ $\gamma < 0$ grow + decay solns; r_1, r_2 have opposite signs

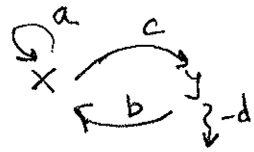
details will follow

$\beta^2 - 4\gamma < 0$

- oscillations
- real part of roots, $\frac{\beta}{2}$ determines if ampl. grows or decays.

$\beta < 0$ ampl. shrink

$\beta > 0$ " grow.



Suppose we are given the rates of the growth, death, etc.

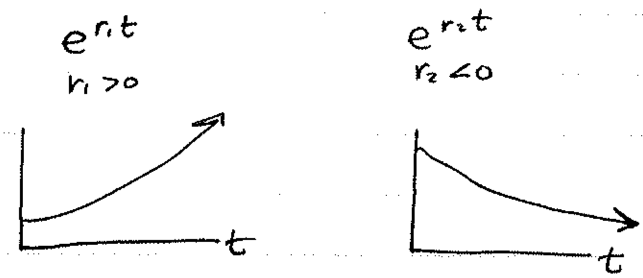
example ② $\underset{a_{11}}{a} = 0.5$ $\underset{a_{12}}{b} = 1$ $\underset{a_{21}}{c} = 1$ $\underset{a_{22}}{d} = -0.1$ t^{-1}

then $\beta = 0.5 - 0.1 = 0.4 > 0$

$\gamma = -0.05 - 1 = -1.05$

$\beta^2 - 4\gamma = 4.36 > 0$

No oscill., some exp growing solns and some exp. decaying solns



So we can tell a lot about the qualitative behavior of the system from its coefficients, even before we find detailed solns.

Remark: what about $x(t)$?

Answ: once we know $y(t)$, we also know $x(t)$

From $*$

$$x(t) = \frac{1}{a_{21}}\frac{dy}{dt} - \frac{a_{22}}{a_{21}} y$$

$$= y_0 \frac{r e^{rt}}{a_{21}} - \frac{a_{22} y_0 e^{rt}}{a_{21}} = y_0 \left(\frac{r - a_{22}}{a_{21}} \right) e^{rt}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{r-a_{22}}{a_{21}} \\ 1 \end{pmatrix} y_0 e^{rt} \quad \text{where} \quad r = \frac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$$

$\beta = a_{11} + a_{22}$

$\gamma = a_{11}a_{22} - a_{12}a_{21}$

How do such solns behave?

- we expect several cases (and we will explore details of such cases in detail).

- $\beta^2 < 4\gamma \Rightarrow$ imaginary roots $\Rightarrow$ cycles and oscillations

  $\beta < 0 \Rightarrow$ negative real parts $e^{-\sigma t}(\sin + \cos)$ (decay)

  $\beta > 0 \Rightarrow$ positive real parts $e^{+\sigma t}(\ldots)$ (growth)

- $\beta^2 > 4\gamma \Rightarrow$ real roots

  $\gamma > 0$ both roots positive $r_1, r_2 > 0$ $e^{r_1 t}, e^{r_2 t}$ growing exp

  $\gamma = 0$ repeated roots te^{rt}

  $\gamma < 0$ ~~but~~ roots of opposite signs $r_1 > 0$ $r_2 < 0$

[Details to follow]

Q: Do we always reduce the system to a single ODE?

Ans: No, not necessarily! In fact, we can use some linear algebra to solve this problem in its original form.

$$\begin{cases} \dfrac{dx}{dt} = a_{11}x + a_{12}y \\ \dfrac{dy}{dt} = a_{21}x + a_{22}y \end{cases} \rightarrow \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = M\vec{x}$$

where $\vec{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ $\quad M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Assume solns in form $\vec{x} = \vec{V}e^{rt}$ where $\vec{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ are constants

Then $\dfrac{d\vec{x}}{dt} = \vec{V} re^{rt}$ so

$$\vec{V}re^{rt} = \underbrace{M \cdot \vec{V}}_{\text{matrix multiplication}} e^{rt}$$

2x2 matrix · 2x1 vector

$e^{rt} \neq 0 \Rightarrow$

$$r\vec{V} = M \cdot \vec{V}$$

Remark: In the book's notation
matrix $M \rightarrow A$
Const. vector $\vec{V} \rightarrow \vec{\xi}$
See pp 390 -- B+D 9th ed ξ Prime

rewrite as

$$M \cdot \vec{V} - r\vec{V} = 0$$

or as: $(M - rI) \cdot \vec{V} = 0$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix

$$\begin{pmatrix} a_{11}-r & a_{12} \\ a_{21} & a_{22}-r \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is just a system of linear algebraic equations in the "unknowns" V_1, V_2. One solution is just $V_1 = V_2 = 0$ (← the uninteresting trivial soln), which is usually unique, is the only soln!

The only way to have nontrivial solns to this algebraic system is if

$$\det (M - rI) = 0 \qquad \text{i.e. } \det \begin{pmatrix} a_{11}-r & a_{12} \\ a_{21} & a_{22}-r \end{pmatrix} = 0$$

This is true only for certain values of r, that we call <u>eigenvalues</u>

The corresponding values of $\vec{v}$ satisfy $M \cdot \vec{v} = r\vec{v}$ are called <u>eigenvectors</u>

Q: How do we find those eigenvalues?

Answ: ① eigenvalues:

$$\det \begin{pmatrix} a_{11}-r & a_{12} \\ a_{21} & a_{22}-r \end{pmatrix} = (a_{11}-r)(a_{22}-r) - a_{12}a_{21} = 0$$

$$a_{11}a_{22} - a_{11}r - a_{22}r - r^2 - a_{12}a_{21} = 0$$

$$r^2 - \underbrace{(a_{11}+a_{22})}_{\beta} r + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{\gamma} = 0$$

$$r^2 - \beta r + \gamma = 0 \qquad \text{where } \beta = a_{11}+a_{22}, \quad \gamma = a_{11}a_{22} - a_{12}a_{21}$$

Note: We have arrived at the same characteristic eqn as we got earlier (when we reduced the system of ODEs to the 2nd order ODE).

Moreover, we now recognize that

$$\beta = a_{11} + a_{22} = \text{Trace of matrix } M = Tr(M)$$

(sum of diagonal elements)

$$\gamma = a_{11}a_{22} - a_{12}a_{21} = \text{determinant of matrix } M = \det(M)$$

eigenvalues are thus $r_{1,2} = \dfrac{+\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$ ←

$\boxed{\text{for sys of 2 ODEs we will have TWO EIGENVALUES}}$

(We again have many cases to consider for the behavior of $e^{r_1 t}$, $e^{r_2 t}$).

Q : How do we find the eigenvectors ?

Suppose $\vec{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is an eigenvector. Then

$$r\vec{V} = M \cdot \vec{V} \quad \Rightarrow \quad \begin{pmatrix} rV_1 \\ rV_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} rV_1 = a_{11}V_1 + a_{12}V_2 \\ rV_2 = a_{21}V_1 + a_{22}V_2 \end{cases}$$

← However, since we are solving this system when $\det(M - rI) = 0$, the two eqns are not linearly independent (i.e. they "duplicate" the information)

So take any <u>one</u> of these, e.g. 2nd eqn:

$$rV_2 = a_{21}V_1 + a_{22}V_2 \quad \text{i.e.} \Rightarrow \quad (r - a_{22})V_2 = a_{21}V_1$$

Let us (arbitrarily) set $V_2 = 1$ and find V_1. Then

$$(r - a_{22}) \cdot 1 = a_{21} V_1 \qquad V_1 = \frac{r - a_{22}}{a_{21}}$$

Thus we have found that each soln is of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} e^{rt} = \begin{pmatrix} \frac{r - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{rt}$$

but there are <u>two</u> of these! One for $r = r_1$, one for $r = r_2$!

The two solns will look like:

$$\begin{pmatrix} \frac{r_1 - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{r_1 t}, \quad \begin{pmatrix} \frac{r_2 - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{r_2 t}$$

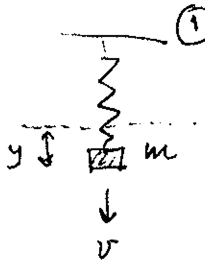
corresponding to eigenvalue r_1 — and to eigenvalue r_2

The general soln will be a LINEAR SUPERPOSITION OF these, i.e.

$$\boxed{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} \frac{r_1 - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} \frac{r_2 - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{r_2 t}}$$

Examples of systems of differential equations

$$\boxed{m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0}$$



(1) SPRING - MASS :

$y(t)$ = displacement (vertical) of mass

$v(t)$ = velocity of mass (by definition, $v(t) = dy/dt$)

equivalent system:

$$\left\{\begin{array}{l} \dfrac{dy}{dt} = v \\ m\dfrac{dv}{dt} + cv + ky = 0 \end{array}\right\} \rightarrow \begin{array}{l} \dfrac{dy}{dt} = v \\ \dfrac{dv}{dt} = -\left(\dfrac{k}{m}\right)y - \left(\dfrac{c}{m}\right)v \end{array}$$

Can be written in matrix form as:

$$\boxed{\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}}_{\text{matrix } M}\begin{pmatrix} y \\ v \end{pmatrix}}$$

Matrix of coeffs: $a_{11} = 0$, $a_{12} = 1$, $a_{21} = -\frac{k}{m}$, $a_{22} = -\frac{c}{m}$

$$\beta = \text{Trace}(M) = 0 - \frac{c}{m} = -\frac{c}{m}$$

$$\gamma = \det(M) = 0\cdot\left(-\frac{c}{m}\right) - (1)\cdot\left(-\frac{k}{m}\right) = \frac{k}{m}$$

Char. eqn : $r^2 - \beta r + \gamma = 0 \quad \leftrightarrow \quad r^2 + \frac{c}{m}r + \frac{k}{m} = 0$

eigenvalues $\quad r_1, r_2 = \dfrac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2} = \dfrac{1}{2}\left(-\dfrac{c}{m} \pm \sqrt{\left(\dfrac{c}{m}\right)^2 - \dfrac{4k}{m}}\right)$

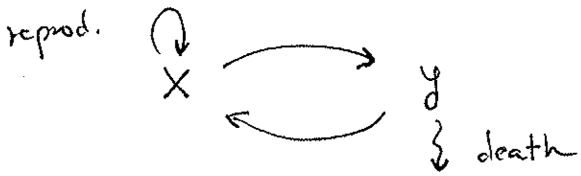
eigenvectors : $\quad \vec{V}_1 = \begin{pmatrix} \dfrac{r_1 - a_{22}}{a_{21}} \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{r_1 + \frac{c}{m}}{-k/m} \\ 1 \end{pmatrix} = \begin{pmatrix} -\dfrac{r_1 m + c}{k} \\ 1 \end{pmatrix}$

$\vec{V}_2 =$ (similarly) .... $\begin{pmatrix} -\dfrac{r_2 m + c}{k} \\ 1 \end{pmatrix}$

gen'l soln:

$$\begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = C_1\vec{V}_1 e^{r_1 t} + C_2\vec{V}_2 e^{r_2 t}$$

TWO - SPECIES SYSTEM



A type of cell can have two forms that can interconvert. One form is reproductive, the other not.

$x(t)$ = dens. of reproducing cells (growth rate 0.5)

$y(t)$ = " " non-reproducing cells (has some death rate $d = 0.1$)

$$\begin{cases} \dfrac{dx}{dt} = 0.5x + y \\ \dfrac{dy}{dt} = x - 0.1y \end{cases} \longrightarrow$$

We can treat this as a system of 2ODES in the two variables.

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{bmatrix} 0.5 & 1 \\ 1 & -0.1 \end{bmatrix}}_{\text{matrix } M}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$M = \begin{bmatrix} 0.5 & 1 \\ 1 & -0.1 \end{bmatrix} \qquad a_{11} = 0.5, \quad a_{12} = 1, \quad a_{21} = 1, \quad a_{22} = -0.1$$

$$\beta = Tr(M) = 0.5 - 0.1 = 0.4$$

$$\gamma = \det(M) = (0.5)(-0.1) - (1)(1) = -0.05 - 1 = -1.05$$

eigenvalues: $r^2 - \beta r + \gamma = 0 \qquad r^2 - 0.4r - 1.05 = 0$

$$r = \frac{0.4 \pm \sqrt{0.16 + 4.2}}{2} = 0.2 \pm \frac{1}{2}\sqrt{4.36}$$

$$r_1 = 1.24 \quad , \quad r_2 = -0.84$$

eigenvectors: $\vec{V}_1 = \begin{pmatrix} -0.8 \\ -0.6 \end{pmatrix} \qquad \vec{V}_2 = \begin{pmatrix} 0.6 \\ -0.8 \end{pmatrix}$

General soln:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} -0.8 \\ -0.6 \end{pmatrix} e^{1.24t} + c_2 \begin{pmatrix} 0.6 \\ -0.8 \end{pmatrix} e^{-0.84t}$$