Nov 17

Sys. of 1st order Linear ODEs and phase planes

$a_{11} = -2, \; a_{12} = 1$
$a_{21} = 1, \; a_{22} = -2$

Example ①

System S1 $\begin{cases} \dfrac{dx}{dt} = -2x + y \\ \dfrac{dy}{dt} = x - 2y \end{cases}$ $\qquad \dfrac{d\vec{x}}{dt} = M\vec{x} \qquad M = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$\beta = \mathrm{Tr}(M) = -4$

$\gamma = \det(M) = 4 - 1 = 3$

$\beta^2 - 4\gamma = 16 - 12 = 4 > 0$ (← already find out we do not expect oscill.)

char. eqn $\quad r^2 - \beta r + \gamma = 0 \qquad r^2 + 4r + 3 = 0$

roots (≡ eigenvalues)

$$r_{1,2} = \frac{-4 \pm \sqrt{4}}{2} = -1, -3$$

eigenvalues:

For $r_1 = -1$ $\qquad \vec{v}_1 = \begin{pmatrix} 1 \\ \frac{r_1 - a_{11}}{a_{12}} \end{pmatrix}$ ← remember, there are various ways to express eigenvectors. (See Nov 15)

$$\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{-1-(-2)}{1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $r_2 = -3$ $\qquad \vec{v}_2 = \begin{pmatrix} 1 \\ \frac{-3-(-2)}{1} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solns: $\vec{x}_1(t) = \vec{v}_1 e^{r_1 t}$

$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$ ← exponential decay

$\vec{x}_2(t) = \vec{v}_2 e^{r_2 t}$

$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$ ← also exponential decay

$= \begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix}$

Q1: Can we express <u>every</u> soln to sys $S1$ as a linear combination ("superposition") of these two solns?

⇛ (Restated question:) is this set of solns a <u>fundamental set</u>?
⇛ (" " ") are $\vec{x}_1$ and $\vec{x}_2$ <u>linearly independent</u>?

→ Remark: recall similar issues in discussing solns to 2nd order ODE (see Sept 22 - 27 lectures)

<u>Answer:</u> Use the idea of <u>Wronskian</u> (properly redefined) to check if $\vec{x}_1, \vec{x}_2$ are linearly indep.

<u>Defn</u> of $\boxed{\text{Wronskian}}$ for system of ODEs:

$$W = \det\left[\vec{x}_1 \quad \vec{x}_2\right]$$

determinant of matrix formed by the two soln vectors side by side

<u>Thm</u>: If $W \neq 0$ then $\vec{x}_1, \vec{x}_2$ form a fundam. set!

Example: for $S1$,

$$W = \det\left[\vec{v}_1 e^{-t} \quad \vec{v}_2 e^{-3t}\right] = \det\begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t}\end{bmatrix}$$

$$= -e^{-t}e^{-3t} - e^{-t}e^{-3t} = -2e^{-t}e^{-3t} = -2e^{-4t} \neq 0$$

So the solns we found are a fundam. set

⇓

The bottom line: We can express every soln to $S1$ as

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-3t}$$

Notes about "special solutions"

Suppose (in above example), initial cond's are as follows:

at time $t=0$

(1) $\vec{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Then we'll find $c_1 = c_2 = 0$ so $\vec{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all t!

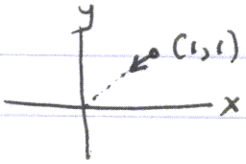
$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a "fixed point" (or steady state) of the system

there is no change from that I.C.

V_1

(2) Suppose $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\Rightarrow$ we'll find $c_1 = 1$, $c_2 = 0$

$$\boxed{\begin{aligned} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ 1 &= c_1 + c_2 \\ 1 &= c_1 - c_2 \\ c_1 &= 1, \; c_2 = 0 \end{aligned}}$$

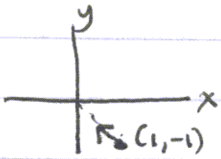


so $\vec{x}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$

in fact, any initial cond that is a scalar multiple of $\vec{V}_1$ will lead to a soln of the form $\vec{x}(t) = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$

V_2

(3) Suppose $\vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ we similarly find $c_1 = 0$, $c_2 = 1$



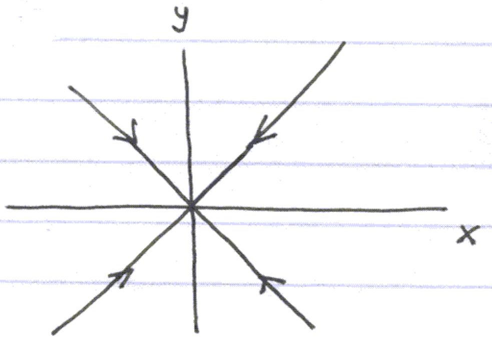
so soln looks like $\vec{x}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$

same idea for any I.C. that is a multiple of V_2

<u>Conclusion</u> (applies more generally to cases of sys. with real eigenvalues)

For any initial conds that lie on directions of the eigenvectors, the solns flow in straight lines to/from the origin.

Q3: How do we connect the soln of a sys. of ODEs to the graphical (phase-plane) behaviour?



Answer

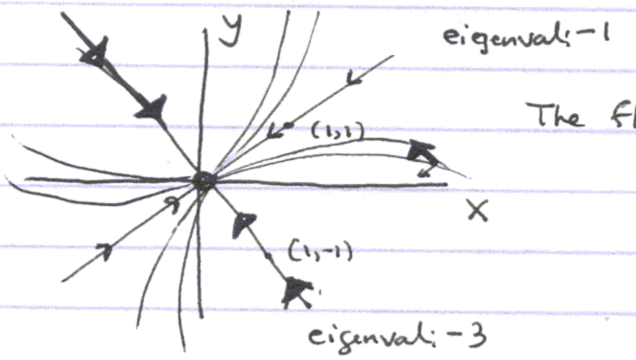
- Along points that are (multiples of) eigen ~~vector~~ directions, solns come in (or go out) in same direction

- −ve (real) eigenvalues ⇒ solns decay i.e. move towards (0,0)

- +ve eigenvalues ⇒ solns grow (move away from (0,0))

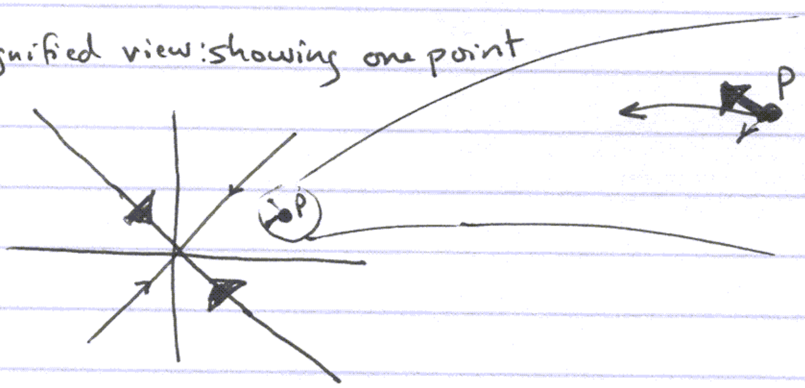
- Large/ magnitude eigenvalues mean rapid motion
- small magnitude eigenvals mean slower motion in xy plane

- This leads to curvature of the trajectories



The flow in the direction with larger eigenvalue (eigenvector) is faster

Magnified view: showing one point



point P feels stronger pull in direction of thick arrow because the eigenvalue is −3 and $|-3| > |-1|$

Other examples

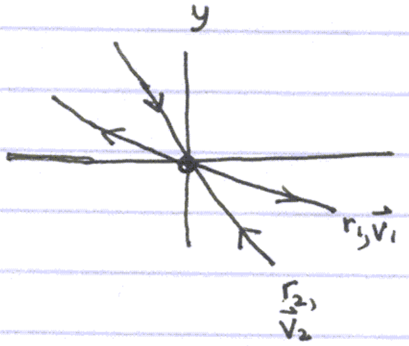
Ex. 2

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 5 & 8 \\ -3 & -5 \end{pmatrix} \vec{x}$$

eigenvalues: $r_1 = 1$ $\quad$ $r_2 = -1$

eigenvectors: $\vec{v}_1 = \begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix}$ $\quad$ $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

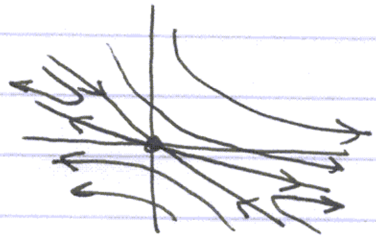
gen'l soln: $\vec{x}(t) = c_1 e^{t} \begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$



This kind of behaviour is called a saddle

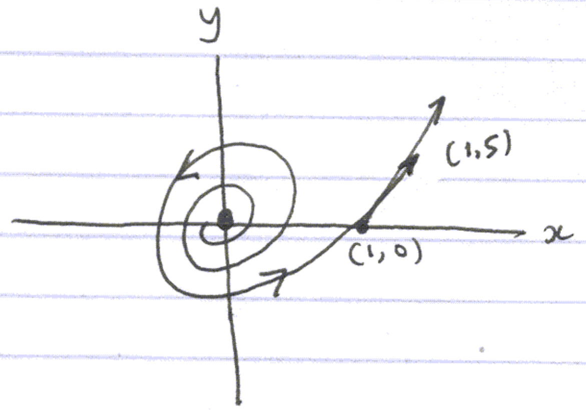
Note solns approach $(0,0)$ along $\vec{v}_2$ and leave $(0,0)$ along $\vec{v}_1$

If we fill in the picture, we get:



Ex 3 $\quad \dfrac{d\vec{x}}{dt} = \begin{pmatrix} 1 & -2 \\ 5 & 1 \end{pmatrix} \vec{x}$ $\qquad$ eigenvalues: $r_{1,2} = 1 \pm \sqrt{10}\, i$

In this case, we do not attempt to draw eigenvectors. We know there will be oscillations of growing amplitude



$e^{\sigma t}$ (cos μt, sin μt etc)

$\sigma = 1$, $\quad \mu = \sqrt{10}$

We can tell if spirals are clockwise or anticlockwise by checking direction of flow at any one point,

e.g. at $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$