Nov 1, 2010

Math 265

Convolutions, Laplace Transforms, and the Transfer function

Let $f(t)$ and $g(t)$ be two functions. Then we define a special kind of product, called a convolution of f and g as follows:

$$f * g = \int_0^t f(t-\tau)\, g(\tau)\, d\tau = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

For the geometric intuition behind this definition, see excerpt from Wikipedia (over). $f*g$ is a kind of weighted average of one function with respect to (a reflected copy of) another function.

Convolutions are applied extensively in digital signal processing, and other applications. We restrict attention to their use in the context of Laplace Transforms.

Properties:

$$f*g = g*f$$

$$f*(g+h) = f*g + f*h$$

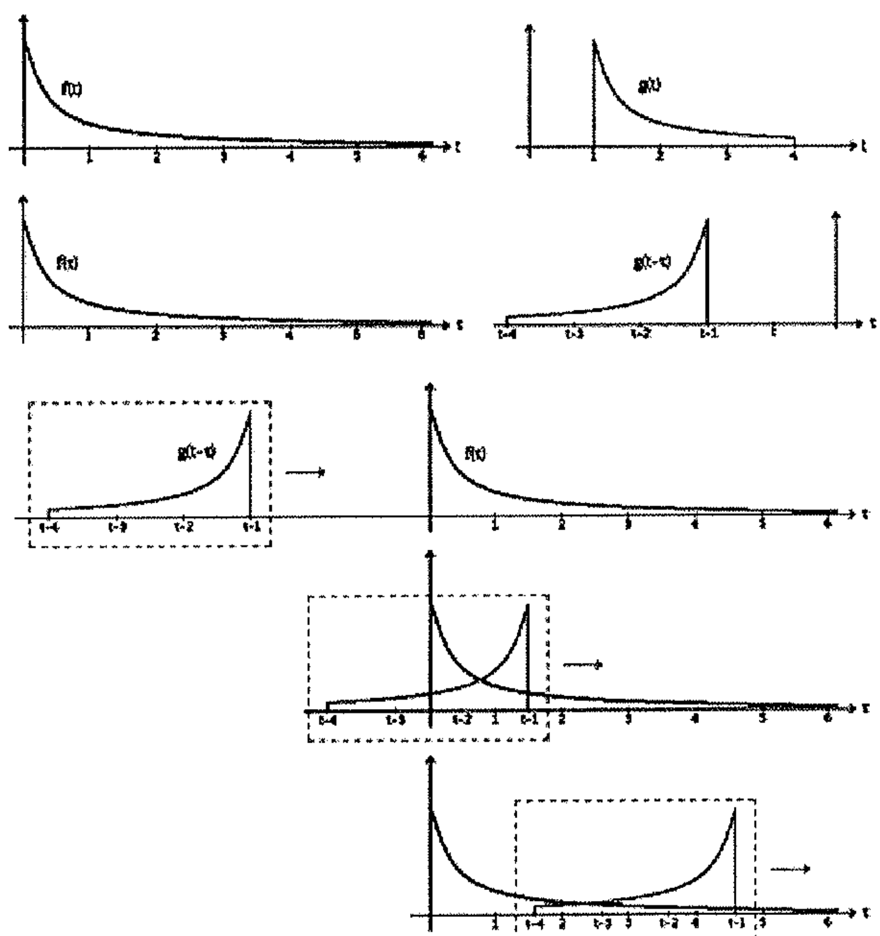
$$f*(g*h) = (f*g)*h$$

See also, a nicer copy of this on a direct link, below the link for this scanned lecture.

http://en.wikipedia.org/wiki/Convolution

1. Express each function in terms of a dummy variable τ.
2. Reflect one of the functions: $g(\tau) \rightarrow g(-\tau)$.
3. Add a time-offset, t, which allows $g(t-\tau)$ to slide along the τ-axis.
4. Start t at $-\infty$ and slide it all the way to $+\infty$. Wherever the two functions intersect, find the integral of their product. In other words, compute a sliding, weighted-average of function $f(\tau)$, where the weighting function is $g(-\tau)$.

The resulting waveform (not shown here) is the convolution of functions f and g. If $f(t)$ is a unit impulse, the result of this process is simply $g(t)$, which is therefore called the impulse response.



Convolution and the Laplace transform

If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$

then $\mathcal{L}^{-1}\{F(s)\,G(s)\} = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$

$$\overset{\text{def}}{=} \quad f * g\,(t)$$ "Convolution of the functions f and g"

<u>Proof:</u> $F(s) = \int_0^\infty e^{-s\xi} f(\xi)\,d\xi$, $G(s) = \int_0^\infty e^{-s\tau} g(\tau)\,d\tau$ by defn.

So their product is:

$$F(s)\,G(s) = \int_0^\infty e^{-s\xi} f(\xi)\,d\xi \cdot \int_0^\infty e^{-s\tau} g(\tau)\,d\tau$$

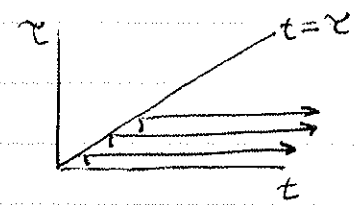
$$= \int_0^\infty \int_0^\infty e^{-s(\xi+\tau)} f(\xi)\,g(\tau)\,d\xi\,d\tau$$

$$= \int_0^\infty g(\tau)\left[\int_0^\infty e^{-s(\xi+\tau)} f(\xi)\,d\xi\right] d\tau$$

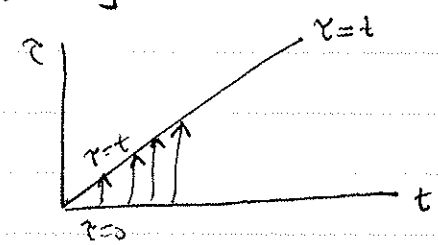
let $\xi = t - \tau$ for fixed τ $\Rightarrow d\xi = dt$

$\xi = 0$ when $t = \tau$
$\xi = \infty$ when $t = \infty$

$$F(s)\,G(s) = \int_{\tau=0}^{\tau=\infty} g(\tau)\left[\int_{t=\tau}^{t=\infty} e^{-st} f(t-\tau)\,dt\right] d\tau$$



$$= \int_0^\infty e^{-st}\left[\int_0^t g(\tau)\,f(t-\tau)\,d\tau\right] dt$$



$$= \mathcal{L}\{g * f\}$$

Computing Convolutions

Examples

① $f(t) = \sin t$ $\qquad f(t-\tau) = \sin(t-\tau)$

$g(t) = \delta(t-1) =$ unit impulse at time 1 $\qquad g(\tau) = \delta(\tau-1)$

$$f*g(t) = \int_0^t \sin(t-\tau)\,\delta(\tau-1)\,d\tau = \sin(t-1)$$

by defn of δ function

(i.e. produces value of $\sin(t-\tau)$ for the single value $\tau = 1$, where δ fn is "concentrated")

② $f(t) = \sin t$

$g(t) = e^t$

$\sin(t-\tau) = \sin(t)\cos(\tau) - \sin(\tau)\cos(t)$

$$f*g(t) = \int_0^t \sin(t-\tau)\,e^\tau\,d\tau = \frac{1}{2}e^\tau\left(\cos(t-\tau) + \sin(t-\tau)\right)\Big|_{\tau=0}^{t}$$

$$= -\frac{1}{2}\cos(t) - \frac{1}{2}\sin(t) + \frac{1}{2}e^t$$

③ $f(t) = \sin t$

$g(t) = \cos t$

$$f*g(t) = \int_0^t \sin(t-\tau)\cos(\tau)\,d\tau = \frac{1}{4}\cos(t-2\tau) + \frac{1}{2}\sin(t)\,\tau\Big|_{\tau=0}^{\tau=t}$$

$$= \frac{1}{2}t\sin t$$

④ $f(t) = \sin t$

$g(t) = t\cos t$

$$f*g(t) = \int_0^t \sin(t-\tau)\cdot\tau\cos(\tau)\,d\tau =$$

poof! pow! Then a miracle happened (Maple)

$$= \frac{1}{4}\left(-\sin(t) + t^2\sin(t) + t\cos(t)\right)$$

(or else.. some long + arduous trig identities and IBP.

Motivation for the usefulness of Convolution Theorem.

Suppose we were asked to solve the I.V.P.

$$\boxed{y'' + y = t\cos t \qquad y(0)=0 \quad y'(0)=0}$$

From previous experience, We expect to see terms such as $t(At+B)(\cos t) + t(Ct+D)(\sin t)$ in such a case

$$s^2F(s) - s\underset{0}{\underset{\shortparallel}{f'(0)}} - \underset{0}{\underset{\shortparallel}{f(0)}} + F(s) = \mathcal{L}\{t\cos t\}$$

$$(s^2+1)F(s) = \mathcal{L}\{t\cos t\}$$

$$\uparrow$$

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$$

$$\mathcal{L}\{t\cos t) = \frac{d}{ds}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2}$$

We obtain the following algebraic formula for $F(s)$

$$F(s) = \frac{s^2-1}{(s^2+1)^3} = \frac{2s}{2s}\,\frac{(s^2-3+2)}{(s^2+1)^3} = \frac{1}{2s}\left[\frac{2s(s^2-3)}{(s^2+1)^3} + \frac{2}{(s^2+1)^3}\right]$$

The problem (always) is how to invert it to find $y(t)$. Looking at the table:

$$\frac{1}{(s^2+1)} \cdot \frac{s^2-1}{(s^2+1)^3} \quad ?$$

too difficult to break into the right fractions as is

We could try to use these facts but this involves a lot of messing around with rational functions.

$$\mathcal{L}\{t\sin t\} = \frac{d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}$$

$$\mathcal{L}\{t\cos t\} = \frac{s^2-1}{(s^2+1)^2}$$

$$\mathcal{L}\{t^2\cos t\} = \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{2s(s^2-3)}{(s^2+1)^3}$$

$$\mathcal{L}\{t^2\sin t\} = \frac{2(3s^2-1)}{(s^2+1)^3}$$

$$F(s) = \frac{1}{3}\cancel{3s^2}\,\frac{s^2+1}{(s^2+1)^3} = \frac{1}{3}\frac{(3s^2-1-2)}{(s^2+1)^3} = \frac{1}{2\cdot 3}\left(\frac{3s^2-1}{s^2+1)^3}\right) - \frac{2}{3}\cdot\frac{1}{(s^2+1)^3}$$

$$= \frac{As+B}{(s^2+1)^3} + \frac{Cs+D}{(s^2+1)^2} + \quad \text{ugly} \;..\; \text{look for another way!}$$

$$\uparrow$$

Motivation for Convolution. Thm

Let us look at such a problem again, more generally and with Convolutions in mind

Solve the IVP $y'' + y = g(t) \qquad y(0)=1 \;,\; y'(0)=1$

We may be interested in various kinds of forcing functions.

$$\left[s^2 F(s) - s\,y(0) - y'(0)\right] + F(s) = G(s) = \mathcal{L}\{g(t)\}$$

$$(s^2+1)F(s) - s - 1 = G(s)$$

$$F(s) = \frac{s+1}{s^2+1} + \frac{G(s)}{s^2+1}$$

this part is due to initial conditions

this part comes from the forcing function and looks like:

Product in Laplace space

looks like $H(s)\cdot G(s)$ where $H(s) = \mathcal{L}\{\sin t\}$

$$= \frac{s}{s^2+1} + \frac{1}{s^2+1} + \left(\frac{1}{s^2+1}\right)\cdot G(s)$$

$$y(t) = \mathcal{L}^{-1}\{F(s)\} = \cos t + \sin t + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot G(s)\right\}}_{\int_0^t \sin(t-\tau)\, g(\tau)\, d\tau}$$

Convolution in the time domain

It can be easier to calculate this integral than to try to invert this term by other methods.

Now we can "easily" apply this to various forcing functions.

Examples: $g(t) = \cos t$

We already computed $\int_0^t \sin(t-\tau)\cos\tau\, d\tau = \frac{1}{2} t \sin t$

so

$$y(t) = \cos t + \sin t + \frac{1}{2} t \sin t \qquad \text{(resonance case)}$$

$g(t) = t\cos t$

left as exercise (see example 4 on the "computing convol" page)

See Boyce + Diprima 9th Ed. pp348–350

Now let us think even more generally (and "philosophically") about our problems

Input $\longrightarrow$ $\boxed{\text{System of interest}}$ $\longrightarrow$ output: $y(t)$

(forcing function $g(t)$)

(has properties determined by bunch of constants)

We want to know how the output $y(t)$ is related to the input $g(t)$.

The "system" could be an LRC circuit, a spring-mass system, a (linear) pendulum, etc. etc. We've studied many systems that satisfy

$$ay'' + by' + cy = g(t) \qquad y(0) = y_0, \; y'(0) = y_0'$$

a, b, c ← properties of the system (e.g. mass, spring const, etc etc.)

$y(0)$, $y'(0)$ ← initial configuration of system

$g(t)$ ← the Input

$y(t)$ ← the desired (prediction for) the output.

By Laplace Tr. method, we can rewrite this I.V.P. as

$$(as^2 + bs + c)F(s) - (as+b)y_0 - ay_0' = G(s) \equiv \mathcal{L}\{g(t)\}$$

or
$$F(s) = \underbrace{\frac{(as+b)y_0 + ay_0'}{(as^2+bs+c)}} + \underbrace{\frac{1}{as^2+bs+c}} \cdot G(s)$$

let us call those $\Phi(s)$ $\Psi(s)$

- $\Phi(s)$ comes from the initial cond's and homog. problem, that is

$$\Phi(s) = \mathcal{L}\{\phi(t)\} \text{ where } \phi(t) \text{ is a soln to}$$

$$ay'' + by' + cy = 0 \qquad y(0) = y_0, \quad y'(0) = y_0'$$

$\Psi(s)$ comes from the forcing fn., i.e. nonhomog probl. but with trivial initial cond's., that is,

- $\Psi(s) = \mathcal{L}\{\psi(t)\}$ where $\psi(t)$ is a soln to

NOTE!!

$$ay'' + by' + cy = g(t) \qquad y(0) = 0, \quad y'(0) = 0$$

- $F(s) = \Phi(s) + \Psi(s)$

- inverting will lead to

$$y(t) = \mathcal{L}^{-1}\{\Phi(s)\} + \mathcal{L}^{-1}\{\Psi(s)\}$$
$$= \phi(t) + \psi(t)$$

Now look at $\Psi(s)$. Recall from last page that

$$\Psi(s) = \frac{1}{as^2 + bs + c} \cdot G(s)$$

call it $H(s)$ ← this is known as the TRANSFER FUNCTION and it depends only on the system properties

$$\Psi(s) = \underbrace{H(s)}_{\text{system properties}} \cdot \underbrace{G(s)}_{\text{input (or forcing function)}}$$

By the convolution theorem

$$\Psi(s) = H(s) \cdot G(s) \iff \psi(t) = \int_0^t h(t-\tau)\, g(\tau)\, d\tau$$

where $h(t) = \mathcal{L}^{-1}\{H(s)\}$

Thus, the part of the 'output' that results from the input $(g(t))$ can be considered as a convolution of $g(t)$ with a special function $h(t)$.

It turns out that $h(t)$ (which, recall, is $\mathcal{L}^{-1}\{H(s)\}$) is the <u>IMPULSE RESPONSE</u> of the system. (i.e. the output obt'd for a unit impulse input).

$ay'' + by' + cy = \delta(t) \qquad y(0) = 0 \quad y'(0) = 0$

$(as^2 + bs + c)F(s) = e^0 \equiv 1$

$$F(s) = \frac{1}{(as^2+bs+c)} \leftarrow \text{Exactly how we defined } H(s)$$

$$= H(s)$$

Thus $y = h(t) = \mathcal{L}^{-1}\{H(s)\}$ is the soln to this impulse-driven system