Mathematics 265 Section: 101 Full Name: ____________________

Midterm Test Nov 10, 2010

Student Number: ____________________

Instructions: There are **6 pages** in this test (including this cover page).

1. **Caution: There may (or may not) be more than one version of this test paper.**
2. Ensure that your *full* name and student number appears on this page. Circle your section number.
3. No calculators, books, notes, or electronic devices of any kind are permitted.
4. Show all your work. Answers not supported by calculations or reasoning will not receive credit. Messy work will not be graded.
5. Five minutes before the end of the test period you will be given a verbal notice. After that time, you must remain seated until all test papers have been collected.
6. When the test period is over, you will be instructed to put away writing implements. Put away all pens and pencils at this point. Continuing to write past this instruction will be considered dishonest behaviour.
7. Please remain seated and pass your test paper down the row to the nearest indicated aisle. Once all the test papers have been collected, you are free to leave.
8. Exposing your test paper, copying from another student's paper, or sharing information about this test constitutes academic dishonesty. Such behaviour may jeopardize your grade on this test, in this course, and your standing at this university.
9. There is a table of Laplace Transforms on p 6 of this test paper.

Question number	Grade	Value
1		18
2		10
3		16
4		16
Total		60

I have read and understood the instructions and agree to abide by them.

Signed: ____________________

Problem 1: Multiple Choice Questions: Circle ONE correct answer (a, b, c, d, or e). There is no partial credit in this question. Illegible or multiple answers will get no credit.

NOTE: In these questions, the notation for step functions includes $u_c(t) = H(t-c)$.

1: The Laplace transform of the function $f(t) = \dfrac{1}{t-5}$ is

(a) e^{-5s}, (b) e^{5s}, (c) $u_5(s)e^{-5s}$, (d) 1, (e) None of these.

Solution: The function $f(t)$ is not of exponential order, since it has a singularity at $t = 5$. Therefore the answer is None of these.

2 Which of the below corresponds to the inverse Laplace transform of $F(s) = \frac{1}{s-1}e^{-3s}$

(a) $H(t-1)e^{t-1}$, (b) $H(t-3)e^{t}$, (c) $H(t-3)e^{-t}$, (d) $H(t-3)e^{t-3}$, (e) $H(t-1)e^{-3(t-1)}$

Solution: The solution is $H(t-3)e^{t-3}$

3 For the differential equation $y'' + 3y' - 2y = \delta(t)$ with initial conditions $y(0) = 0, y'(0) = 0$, we find that $F(s) = \mathcal{L}\{y(t)\}$ is which of the following functions?

(a) $\displaystyle\int_0^t y(t-\tau)\delta(\tau)d\tau$ (b) $\dfrac{e^s}{s^2+3s-2}$, (c) $\dfrac{s}{s^2+3s-2}$, (d) $\dfrac{e^{2s}}{s^2+3s-2}$, (e) $\dfrac{1}{s^2+3s-2}$,

Solution: The answer is $F(s) = \dfrac{1}{s^2+3s-2}$, also called the *Transfer function.*

4: A "pirate-ship" ride at an amusement park is driven by a motor so that the vertical displacement of the "ship" satisfies $\alpha z'' + \beta z' + \frac{1}{\gamma}z = 10\cos(\omega t)$, where α, β, γ are manufacturer's specifications for the ride, and ω is the driving frequency of the motor powering the ride. Suppose that $\alpha = 12, \gamma = 3$ are fixed. For which of the following settings will the displacement of the ride have the greatest amplitude?

(a) $\beta = 0.01, \omega = 4$, (b) $\beta = 0.01, \omega = 1/6$, (c) $\beta = 1, \omega = \sqrt{15}/24$, (d) $\beta = 1, \omega = 1/6$, (e) $\beta = 2, \omega = \sqrt{3}/12$

Solution: The natural frequency of the (undamped) machine would be $\omega_0 = \frac{\sqrt{4\alpha/\gamma}}{2\alpha} = \frac{1}{\sqrt{\alpha\gamma}}$. The oscillation amplitude is greater the smaller the damping and the closer is the driving frequency to the natural frequency. Here β acts as the damping, so to minimize it, we chose the setting $\beta = 0.01$. The natural frequency for $\alpha = 12, \gamma = 3$ is $\omega_0 = \frac{1}{\sqrt{\alpha\gamma}} = \frac{1}{\sqrt{12\cdot 3}} = \frac{1}{6}$.

Hence the answer is $\beta = 0.01, \omega = 1/6$.

5: Consider the system of first order ODEs given below.

$$\begin{aligned}\frac{dx}{dt} &= -x+2y,\\ \frac{dy}{dt} &= -2x-y.\end{aligned}$$

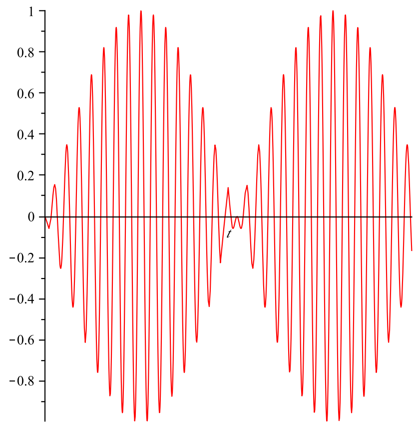
Which of these answers best described the behaviour of the system?
(a) Growing exponential behaviour
(b) Decaying exponential behaviour
(c) Oscillations with decaying amplitude
(d) Oscillations with growing amplitude
(e) None of the above

Solution: The matrix of coefficients has β =Tr(M)=(-1-1)=-2 and γ =det(M)=(1+4)=5. The eigenvalues satisfy $r^2 - \beta r + \gamma = 0$ so

$$r_{1,2} = \frac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2} = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i.$$

The real part is negative and the roots are complex, so we expect to see oscillations with decaying amplitude

6: Consider the differential equation $y'' + 9y = E_0 \cos(\omega t)$. The following behaviour is observed when the driving frequency is $\omega = 3.2$. What is the frequency (radians/time) of the envelope of the oscillations and of the oscillations themselves?

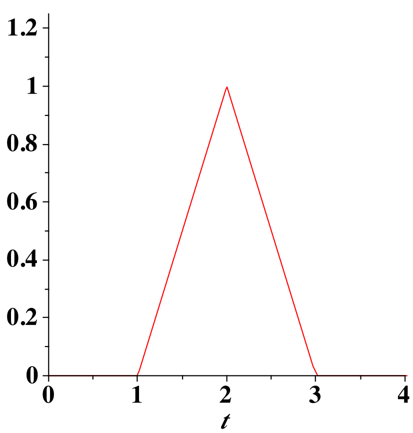


(a) Envelope frequency= 3.1, oscillation frequency 0.1
(b) Envelope frequency= 0.1, oscillation frequency 3.1
(c) Envelope frequency= 0.2, oscillation frequency 6.2
(d) Envelope frequency= 6.2, oscillation frequency 0.2
(e) Envelope frequency= 9/3.2, oscillation frequency 9·3.2

Solution: The solution can be written in the form of $y(t) = C \sin\left(\dfrac{\omega_0 + \omega}{2} t\right) \cdot \sin\left(\dfrac{\omega_0 - \omega}{2} t\right)$ so the envelope frequency (lower of the two) is $\frac{\omega_0 - \omega}{2} = \frac{3.2-3}{2} = 0.1$ and the cycles themselves have frequency $\frac{\omega_0+\omega}{2} = \frac{3.2+3}{2} = 3.1$

Problem 2: Consider the function shown in the figure. Answer (i) and (ii).

(i) Find the Laplace Transform of this function, $\mathcal{L}\{f(t)\}$.



$\mathcal{L}\{f(t)\} =$ ____

(ii) If the same function is extended so that it is periodic, with period $T = 4$, what would be the Laplace transform $\mathcal{L}\{f_p(t)\}$ of the periodic function $f_p(t)$?

$\mathcal{L}\{f_p(t)\} =$ ____

Solution:

(i) The function can be written as:

$$f(t) = [\text{Heaviside}(t-1) - \text{Heaviside}(t-2)] \cdot (t-1) + [\text{Heaviside}(t-2) - \text{Heaviside}(t-3)] \cdot (3-t)$$

also correct is

$$f(t) = [u_1(t) - u_2(t)] \cdot (t-1) + [u_2(t) - u_3(t)] \cdot (3-t)$$

We rewrite it in the form

$f(t) = u_1(t) \cdot (t-1) - u_2(t) \cdot (t-1) + u_2(t) \cdot (3-t) - u_3(t) \cdot (3-t)$

$f(t) = u_1(t) \cdot (t-1) - u_2(t) \cdot (t-2+1) + u_2(t) \cdot (1+2-t) - u_3(t) \cdot (3-t)$

$f(t) = u_1(t) \cdot (t-1) - u_2(t) \cdot (t-2) + u_2(t) \cdot (2-t) - u_3(t) \cdot (3-t)$

$f(t) = u_1(t) \cdot (t-1) - 2u_2(t) \cdot (t-2) - u_3(t) \cdot (3-t)$

In this form we see that its Laplace transform is

$$\mathcal{L}\{f(t)\} = \frac{\mathrm{e}^{-s} - 2\,\mathrm{e}^{-2\,s} + \mathrm{e}^{-3\,s}}{s^2}$$

(ii) For $f_p(t)$ the periodic extension of the above function, with period 4,

$$\mathcal{L}\{f_p(t)\} = \frac{1}{1-e^{-4s}} \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-4s}} \left(\frac{\mathrm{e}^{-s} - 2\,\mathrm{e}^{-2\,s} + \mathrm{e}^{-3\,s}}{s^2} \right)$$

Problem 3: Short Answer Questions

Check your answer carefully, as there are no part marks for right method(s) in this question.

(A) Find the inverse Laplace transform of the following functions:

(a) $F(s) = \dfrac{9}{s^4}$

$\mathcal{L}^{-1}\{F(s)\} =$ ____

(b) $F(s) = \dfrac{2}{(s-2)(s-1)}$

$\mathcal{L}^{-1}\{F(s)\} =$ ____

(c) $F(s) = \dfrac{s}{(s^2+4s+9)}$

$\mathcal{L}^{-1}\{F(s)\} =$ ____

Solution:

(a) $F(s) = \dfrac{9}{s^4} = \dfrac{9}{3!}\dfrac{3!}{s^4}$ The inverse transform is $f(t) = \frac{3}{2}t^3$

(b) The partial fraction expansion of $F(s)$ is

$F(s) = -\dfrac{2}{(s-1)} + \dfrac{2}{(s-2)}$ and the inverse Laplace transform is $f(t) = 2e^{2t} - 2e^t$.

(c) Completing the square and some algebra leads to

$$F(s) = \frac{s}{(s^2+4s+4)+9-4} = \frac{s}{(s+2)^2+5} = \frac{(s+2)-2}{(s+2)^2+5} = \frac{(s+2)}{(s+2)^2+5} - \frac{2}{(s+2)^2+5}$$

$$F(s) = \frac{(s+2)}{(s+2)^2+5} - \frac{2}{\sqrt{5}}\frac{\sqrt{5}}{(s+2)^2+5} = \left(\frac{s}{s^2+5}\right)\Bigg|_{s\to s+2} - \frac{2}{\sqrt{5}}\left(\frac{\sqrt{5}}{s^2+5}\right)\Bigg|_{s\to s+2}$$

We see that this corresponds to cosine and sine with the shift leading to an exponential so the inverse transform is

$$f(t) = e^{-2t}\left(\cos(\sqrt{5}t) - 2\frac{\sqrt{5}}{5}\sin(\sqrt{5}t)\right)$$

(B) Consider the ODE

$$y'' + 4y = t\cos(2t)$$

Using the Method of Undetermined Coefficients, what would be the form of the particular solution to this equation? **NOTE: Do not solve the equation and do not find the coefficients.**

$y_p(t) =$ ____

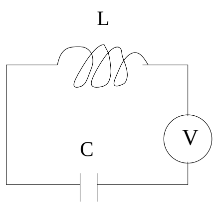
Solution: This is exactly like problem 2D in HW4. The frequency of the unforced system is $\omega = 2$. To avoid duplicating any part of the solution to the homogeneous system, we have to use the guess

$$y_p(t) = t\left[(At + B)\cos(2t) + (Ct + D)\sin(2t)\right]$$

Problem 4: In the Figure shown here, $L = 5$ is inductance and $C = 0.2$ is capacitance in the circuit. The charge on the capacitor satisfies

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = V(t), \quad q(0) = 0, q'(0) = 0. \tag{1}$$

(a) A unit impulse voltage is applied at $t = 0$. You may assume that this impulse is represented by the Dirac delta function. Solve Eqn. (1) to find the charge $q(t)$ at $t > 0$.



$q(t) =$ ____

(b) What would be the current $I(t)$ in the inductor in this same circuit with the same applied voltage?

$I(t) =$ ____

(c) Consider the functions $g(t) = t$ and $h(t) = \sin(t)$. Compute the convolution $g * h$.
[Hint: A useful fact is: $\int x\sin(x)dx = \sin(x) - x\cos(x)$.]

$g * h =$ ____

Problem 4 is continued next page → .. →

Problem 4 Cont'd

(d) A voltage of the form $V(t) = t$ is now applied to the circuit. Use the Laplace transform method to solve Eqn. (1) with this new time-dependent voltage. (You may find the result in (c) useful.)

Solution to Problem 4:

(a)

$$\begin{aligned}
L\frac{d^2q}{dt^2} + \frac{1}{C}q &= \delta(t), \\
L(s^2F(s) - sq(0) - q'(0)) + \frac{1}{C}F(s) &= \mathcal{L}\{\delta(t)\} = 1 \\
(s^2 + \frac{1}{LC})F(s) &= \frac{1}{L} \\
F(s) &= \frac{1}{L}\frac{1}{(s^2 + \frac{1}{LC})} \\
&= \frac{1}{L}\frac{1}{(s^2 + \omega^2)} \\
&= \frac{1}{L\omega}\frac{\omega}{(s^2 + \omega^2)}
\end{aligned}$$

Since $L = 5$, $C = 0.2$ we have that $\omega = 1/\sqrt{LC} = 1$ so

$$F(s) = \frac{1}{5}\frac{1}{(s^2 + 1)}$$

Now inverting this leads to

$$q(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{L\omega}\sin(\omega t) = \sqrt{\frac{C}{L}}\sin\left(\frac{1}{\sqrt{LC}}t\right) = \frac{1}{5}\sin(t)$$

(b) The current is $I(t) = dq/dt$ so

$$I(t) = \sqrt{\frac{C}{L}}\frac{1}{\sqrt{LC}}\cos\left(\frac{1}{\sqrt{LC}}t\right) = \frac{1}{L}\cos\left(\frac{1}{\sqrt{LC}}t\right) = \frac{1}{5}\cos(t)$$

(c) $g(t) = t$ and $h(t) = \sin(t)$ so the convolution is

$$g * h(t) = \int_0^t (t - \tau)\sin(\tau)d\tau = t\int_0^t \sin(\tau)d\tau - \int_0^t \tau\sin(\tau)d\tau$$

$g*h(t) = -t(\cos(t) - \cos(0)) - (\sin(t) - t\cos(t)) + (\sin(0) - 0\cos(0)) = t(1 - \cos(t) + \cos(t)) - \sin(t) = t - \sin(t).$

[where we have used $\int x\sin(x)dx = \sin(x) - x\cos(x)$.]

(d) The new equation and its solution are:

$$\begin{aligned}
L\frac{d^2q}{dt^2} + \frac{1}{C}q &= t, \\
(s^2 + \frac{1}{LC})F(s) &= \mathcal{L}\{t\} = \frac{1}{s^2} \\
F(s) &= \frac{1}{L}\frac{1}{(s^2 + \frac{1}{LC})} \cdot \frac{1}{s^2} \\
&= \frac{1}{L\omega}\frac{\omega}{(s^2 + \omega^2)} \cdot \frac{1}{s^2}
\end{aligned}$$

Again, since $\omega = 1$ we have that

$$F(s) = \frac{1}{L} \frac{1}{(s^2+1)} \cdot \frac{1}{s^2}$$

The final result is a product of two Laplace transforms, i.e.

$$F(s) = \frac{1}{L} H(s) \cdot G(s), \quad \text{where} \quad H(s) = \mathcal{L}\{\sin(t)\}, \quad G(s) = \mathcal{L}\{t\}$$

By the convolution theorem,

$$q(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{L} g(t) * h(t)$$

where $g(t) = t, h(t) = \sin(t)$, but $\omega = 1$ so $q(t) = \frac{1}{L}\int_0^t (t-\tau)\sin(\tau)d\tau = \frac{1}{L}(t - \sin(t)) = \frac{1}{5}(t - \sin(t))$

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$\delta(t-c)$	e^{-cs}
$e^{ct}f(t)$	$F(s-c)$
$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$