Solns to HW 7

Solve the following systems

(1) $\frac{d\vec{x}}{dt} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \vec{x}$ $\qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Soln: $\beta = 10 \quad \gamma = 29 \qquad r_{1,2} = 5 \pm 2i \qquad \vec{V}_{1,2} = \begin{pmatrix} 1 \\ 1 \pm 2i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\beta^2 - 4\gamma = 100 - 116 = -16 < 0$

gen'l soln:

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \left[C_1 \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} + C_2 \begin{pmatrix} \sin 2t \\ -2\cos 2t + \sin 2t \end{pmatrix} \right] e^{5t}$$

$$\vec{x}(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} C_1 \\ C_1 - 2C_2 \end{bmatrix} \qquad \begin{array}{l} C_1 = 1 \\ C_2 = C_1/2 = 1/2 \end{array}$$

$$\vec{x}(t) = \left[1 \cdot \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \sin 2t \\ -2\cos 2t + \sin 2t \end{pmatrix} \right] e^{5t}$$

$$= \begin{bmatrix} \cos 2t + \frac{1}{2}\sin 2t \\ \frac{5}{2} \sin 2t \end{bmatrix} e^{5t}$$

(2) $\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \vec{x}$ $\qquad \vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Soln: $\beta = 4 \quad \gamma = 3 - 15 = -12$

$\beta^2 - 4\gamma = 16 + 48 = 64 \qquad r_{1,2} = -2, 6$

for $r_1 = -2$: $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\qquad$ for $r_2 = 6$: $\vec{v}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

genl soln: $x(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$

at $t=0$ $\quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + 3c_2 \\ -c_1 + 5c_2 \end{pmatrix}$ $\qquad c_1 = -3c_2$

$1 = -c_1 + 5c_2 = 8c_2$

$c_2 = 1/8, \quad c_1 = -3/8$

$$\vec{x}(t) = -\frac{3}{8} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \frac{1}{8} \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

Solve

(3)

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \vec{x}$$

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenvalues: $r_1 = -1$, $r_2 = 4$

eigenvectors: $\vec{V}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{V}_2 = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$

gen l' soln: $\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} e^{4t}$

at $t=0$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ -c_1 + 2/3\, c_2 \end{pmatrix} \Rightarrow$ $2 = c_2 + 2/3\, c_2 = \frac{5}{3} c_2$

$c_2 = 6/5$

$c_1 = 1 - c_2 = -1/5$

$$\vec{x}(t) = -\frac{1}{5} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \frac{6}{5} \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} e^{4t}$$

(4)

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \vec{x}$$

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

eigenvalues: $r = \pm 2i$

eigenvectors: $\vec{V} = \begin{pmatrix} 8 \\ -2 \pm 2i \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix} \pm \begin{pmatrix} 0 \\ -2 \end{pmatrix} i$

$= \vec{a} \pm \vec{b} i$

gen l' soln:

$$\vec{x}(t) = c_1 \left[\begin{pmatrix} 8 \\ -2 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin 2t \right] + c_2 \left[\begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 2t - \begin{pmatrix} 8 \\ -2 \end{pmatrix} \sin 2t \right]$$

IC's:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \left[\begin{pmatrix} 8 \\ -2 \end{pmatrix} \right] + c_2 \left[\begin{pmatrix} 0 \\ -2 \end{pmatrix} \right] = \begin{bmatrix} 8c_1 \\ -2c_1 - 2c_2 \end{bmatrix}$$

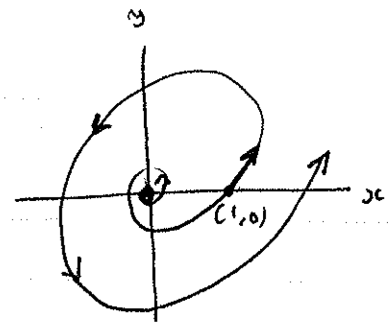
$\Rightarrow c_1 = 1/8$

$c_2 = -c_1 = -1/8$

$$\vec{x}(t) = \frac{1}{8} \left[\begin{pmatrix} 8 \\ -2 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin 2t \right] - \frac{1}{8} \left[\begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 2t - \begin{pmatrix} 8 \\ -2 \end{pmatrix} \sin 2t \right]$$

$$\vec{x}(t) = \begin{bmatrix} \cos 2t + \sin 2t \\ -\frac{1}{2} \sin 2t \end{bmatrix}$$

1

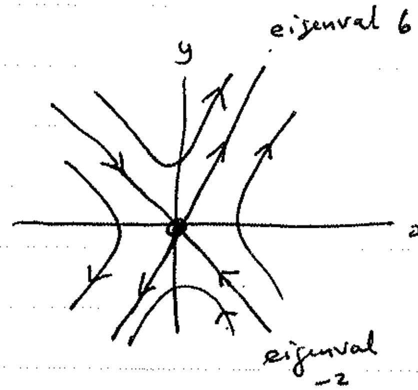


Solns are oscillatory, with increasing amplitude

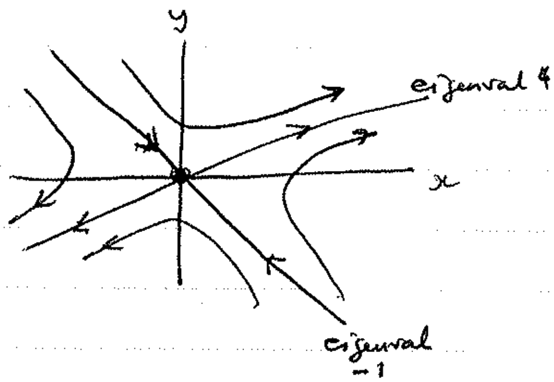
To get which direction spirals go, plug in any one point, e.g. at

$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{d\vec{x}}{dt} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

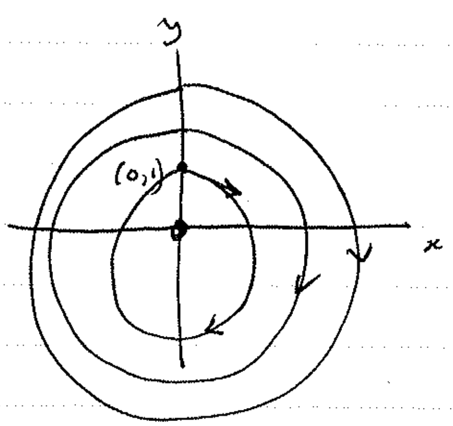
2



3



4



Solns are neutral cycles (amplitude does not grow nor shrink)

to get direction of cycle: plug in any I.C. e.g. at $\vec{x} = (0,1)$

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$$

⑤

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \vec{x} \qquad x(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

eigenvalues: $r_1 = 1 \quad r_2 = 2$ (since matrix is diagonal)

eigenvectors: $(M - rI) \cdot \vec{v} = \vec{0}$

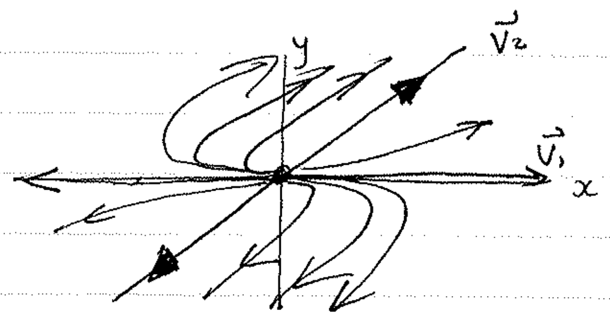
$$\begin{pmatrix} 1-r & 3 \\ 0 & 2-r \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} (1-r)\, v_1 + 3 v_2 &= 0 \\ (2-r) v_2 &= 0 \end{aligned}$$

let $v_1 = 1$ then $v_2 = \frac{1-r}{3}$

so for eigenvalue $r_1 = 1$ $\quad \vec{V}_1 = \begin{pmatrix} 1 \\ \frac{1-r}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

for $\quad r_2 = 2 \quad \vec{V}_2 = \begin{pmatrix} 1 \\ \frac{1-r}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix}$ ← direction of largest magnitude eigenvalue (fastest flow)

Soln: $\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} e^{2t}$



eigenvalues both positive ⟹ flow away from origin, and most rapid in direction of $\vec{V}_2$

IC's: at $t=0$ $\quad \vec{X}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} \qquad \begin{aligned} -1 &= c_1 + c_2 \\ 1 &= \tfrac{1}{3} c_2 \end{aligned}$

$$\Rightarrow \quad c_2 = 3 \qquad c_1 = -1 - c_2 = -4$$

$$\vec{x}(t) = -4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + 3 \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} e^{2t}$$