Example: How to use Laplace Tr. and Convolution Thm to solve:

$$\boxed{y'' - 2y' - 3y = t \qquad y(0) = 0 \quad y'(0) = 0}$$

$$\left(s^2F(s) - s\underset{0}{y(0)} - \underset{0}{y'(0)}\right) - 2\left(sF(s) - y(0)\right) - 3F(s) = \mathcal{L}\{t\}$$

$$(s^2 - 2s - 3)\,F(s) = \mathcal{L}\{t\}$$

$$F(s) = \frac{1}{(s^2 - 2s - 3)}\,\mathcal{L}\{t\}$$

$$= \frac{1}{(s-3)(s+1)}\,\mathcal{L}\{t\}$$

Partial Frac

$$\frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} \Rightarrow A(s+1) + B(s-3) = 1$$

$$s \to -1 \Rightarrow B(-4) = 1 \qquad B = -1/4$$

$$s \to 3 \Rightarrow A(4) = 1 \qquad A = 1/4$$

$$F(s) = \frac{1}{4}\left(\frac{1}{s-3} - \frac{1}{s+1}\right)\mathcal{L}\{t\}$$

$$= \frac{1}{4}\left(\mathcal{L}\{e^{3t}\} - \mathcal{L}\{e^{-t}\}\right)\mathcal{L}\{t\}$$

$$= \frac{1}{4}\left(\mathcal{L}\{e^{3t}\}\mathcal{L}\{t\} - \mathcal{L}\{e^{-t}\}\mathcal{L}\{t\}\right)$$

Use convol. thm to invert

$$y(t) = \frac{1}{4}\left[\int_0^t (t-\tau)\,e^{3\tau}\,d\tau - \int_0^t (t-\tau)\,e^{-\tau}\,d\tau\right]$$

$$= \frac{1}{4}\left[\underbrace{t\int_0^t e^{3\tau}\,d\tau}_{1} - \underbrace{\int_0^t \tau e^{3\tau}\,d\tau}_{2} - \underbrace{t\int_0^t e^{-\tau}\,d\tau}_{3} + \underbrace{\int_0^t \tau e^{-\tau}\,d\tau}_{4}\right]$$

We need to compute one integration by parts.

p2

$u = \tau, \; dv = e^{a\tau} d\tau, \; du = d\tau \quad v = \frac{e^{a\tau}}{a}$

$$\text{IBP} \quad \int_0^t \tau e^{a\tau} d\tau = \left[\frac{1}{a}\tau e^{a\tau} - \frac{1}{a^2} e^{a\tau}\right]\Big|_0^t$$

$$= \frac{1}{a} t e^{at} - \frac{1}{a^2} e^{at} + \frac{1}{a^2}$$

$$= \frac{1}{a} e^{at}\left(t - \frac{1}{a}\right) + \frac{1}{a^2}$$

$$y(t) = \frac{1}{4}\left[\overbrace{t\left(-\frac{1}{3} + \cancel{\frac{1}{3}e^{3t}}\right)}^{1} - \overbrace{\left(\frac{1}{9} - \frac{1}{9}e^{3t} + \cancel{\frac{t}{3}e^{3t}}\right)}^{2} - \overbrace{t\left(1 - \cancel{e^{-t}}\right)}^{3} + \overbrace{\left(1 - e^{-t} - \cancel{te^{-t}}\right)}^{4}\right]$$

Collect terms, cancel some terms to get:

$$y(t) = -\frac{1}{3}t + \frac{1}{36}e^{3t} - \frac{1}{4}e^{-t} + \frac{2}{9}$$

Undetermined Coefficients

$y'' - 2y' - 3y = t \qquad y(0) = 0 \qquad y'(0) = 0$

hom eqn $\quad y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0 \qquad (r-3)(r+1) = 0 \qquad r = 3,\ r = -1$

$y_{hom}(t) = C_1 e^{-t} + C_2 e^{3t}$

$$\left.\begin{aligned} y_p(t) &= At + B \\ y'_p(t) &= A \\ y''_p(t) &= 0 \end{aligned}\right\} \rightarrow \quad -2A - 3(At+B) = t$$

$$\Rightarrow \quad -3A = 1 \qquad A = -1/3$$

$$-2A \cancel{+B} - 3B = 0$$

$$B = -\frac{2}{3}\cdot A = \frac{2}{9}$$

$y_p(t) = -\frac{1}{3}t + \frac{2}{9}$

$y(t) = y_{hom}(t) + y_p(t) = C_1 e^{-t} + C_2 e^{3t} - \frac{1}{3}t + \frac{2}{9}$

$y(0) = 0 \quad \Rightarrow \quad C_1 + C_2 + \frac{2}{9} = 0$

$y'(0) = 0 \quad \Rightarrow \quad -C_1 + 3C_2 - \frac{1}{3} = 0$

$$4C_2 + \frac{2}{9} - \frac{3}{9} = 0$$

$$C_2 = \frac{1}{4}\cdot\frac{1}{9} = \frac{1}{36}$$

$$C_1 = -C_2 - \frac{2}{9} = -\frac{1}{9}\left(\frac{1}{4} + 2\right)$$

$$= \frac{-1}{4}$$

$$\boxed{y(t) = -\frac{1}{4}e^{-t} + \frac{1}{36}e^{3t} + \frac{2}{9} - \frac{1}{3}t}$$

Now consider the revised problem with new driving force and new I.C.'s

$$\boxed{y'' - 2y' - 3y = e^{-t} \qquad y(0) = 0 \quad y'(0) = 1}$$

$$(s^2 - 2s - 3)\,F(s) - 1 = \mathcal{L}\{e^{-t}\}$$

$$F(s) = \frac{1}{(s-3)(s+1)} + \frac{1}{(s-3)(s+1)}\,\mathcal{L}\{e^{-t}\}$$

we don't need to repeat all the work! We already know that part. fract $\Rightarrow$

$$F(s) = \frac{1}{4}\left(\frac{1}{s-3} - \frac{1}{s+1}\right) + \underbrace{\frac{1}{4}\left(\frac{1}{s-3} - \frac{1}{s+1}\right)\mathcal{L}\{e^{-t}\}}_{\text{convolution}}$$

Inverting:

$$y(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{4}\left(e^{3t} - e^{-t}\right) + \frac{1}{4}\left(e^{3t} - e^{-t}\right) * \left(e^{-t}\right)$$

$$= \quad '' \quad '' \quad + \frac{1}{4}\underbrace{\int_0^t \left(e^{3(t-\tau)} - e^{-(t-\tau)}\right)e^{-\tau}\,d\tau}_{\text{convolution integral}}$$

$$= \quad '' \quad '' \quad + \frac{1}{4}\left[e^{3t}\int_0^t e^{-3\tau-\tau}\,d\tau - e^{-t}\int_0^t \underbrace{\cancel{e^{\tau}}\cancel{e^{-\tau}}}_{1}\,d\tau\right]$$

$$= \frac{1}{4}\left(e^{3t} - e^{-t}\right) + \frac{1}{4}\left(\frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} - te^{-t}\right)$$

$$= -\frac{5}{16}e^{-t} + \frac{5}{16}e^{3t} - \frac{1}{4}te^{-t}$$