It is an extremely useful thing to have knowledge of the true origins of memorable discoveries ... It is not so much that thereby history may attribute to each man his own discoveries and that others should be encouraged to earn like commendation, as that the art of making discoveries should be extended by considering noteworthy examples of it.

Leibniz (from the *Historia et Origo Calculi Differentialis*, translated by J. M. Child)

The L-group

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In the late 1960's, Robert Langlands introduced a number of ideas to the theory of automorphic forms and formulated a number of conjectures which gave the theory a new focus. I was a colleague of his at this time, and a good deal of my professional energy since then has been directed to problems posed by him. Thus it was not entirely inappropriate that when I was invited to this conference, Miyake suggested that I say something about those long gone years. I was rather reluctant to do this, and for several reasons. The most important one is that, unlike other mathematicians who have contributed to class field theory and whose work has been discussed at this conference—such as Weber, Takagi, Hasse, or Artin—Langlands himself is still very much alive, and can very well speak for himself. Indeed, in recent years he has shown himself quite willing to discuss his work on automorphic forms in an historical context. A second reason for hesitation on my part was that although my own professional life has practically coincided with that of Langlands' principal conjectures about automorphic forms, and although I have been both a professional and a personal friend of his for that period, my own contributions to the subject have been perhaps of too technical a nature to be of sufficiently general interest to talk about at this conference. A third reason was that if I really were to tell you something new and of historical interest, I would most of all want to be able to refer to correspondence of Langlands during the late 1960's, which has up to now been available only to a few specialists, and details of which I could hardly include in a talk of my own.

However, last summer Langlands and I began a project which caused to me think again about Miyake's suggestion. With the assistance of many other people, we have begun to collaborate in publishing Langlands' collected works on the Internet. This is in many ways an ideal form of publication for something like this. For one thing, much of Langlands' work was first if not exclusively presented in unpublished correspondence and monographs hitherto not easily accessible. My original idea was simply to scan this material electronically for presentation in crude digital format. But Langlands was more ambitious. Currently several of the staff at the Institute for Advanced Study are retyping in TeX not only the unpublished stuff, but in addition many of the published papers and books, for free distribution in electronic format. What we have done so far is now available at the Internet site `http://sunsite.ubc.ca/DigitalMathArchive/Langlands`

The site itself is one of several Internet sites partially sponsored by Sun Microsystems. The original idea for these sites was to make software easily and freely available to the public, but the proposal UBC made to Sun extends the concept of software to include a wide range of mathematical material.

At the moment I write this (July, 1998), what is on line includes

- A letter to André Weil from January 1967
- A letter to Roger Godement from May, 1967
- A letter to J-P. Serre from December, 1967
- *Euler products*, originally published as a booklet by Yale University Press
- 'Problems in the theory of automorphic forms', contained in volume #170 of the Lecture Notes in Mathematics
- 'A bit of number theory', notes from a lecture given in the early 1970's at the University of Toronto

Before the end of this summer of 1998, we will probably have also, among smaller items, the book *Automorphic forms on* $GL(2)$ by Jacquet and Langlands, originally volume #114 of the Lecture Notes in Mathematics; the booklet *Les débuts de la formule de trace stable*, originally published by the University of Paris; and the preprint distributed by Yale University on Artin L-functions and local L-factors, which has seen only extremely limited distribution.

One of the potential problems we expected to cause us some trouble was that of copyright ownership. But I am pleased to say that so far none of the original publishers has offered any obstacle at all to our project. Considering current controversies over these matters, I would like to say that in my opinion the only copyright policy regarding research publication which makes any sense from the overall perspective of the research community is one under which control automatically reverts to an author after, say, at most three or four years.

The ultimate format of the collection has probably not yet been found, but at the moment each item is accompanied by a few editorial remarks as well as comments by Langlands himself. The papers themselves can be retrieved in any of several electronic formats produced from TEX files. Nor is it entirely clear—at least to me—what the final fate of the collection will be, but the advantage of the way in which the project is being carried out is that things will be made available as soon as possible, even if the first versions might be somewhat different from the final ones. My own contribution is essentially editorial, although I and one of my colleagues at UBC are also responsible for technical matters. I would like to point out that this manner of publication is the ideal one in many situations, and that if anyone would like to know exactly what sort of technical effort it involves, I will be happy to try to answer questions.

In the rest of this paper I will recall in modern terms the principal concepts introduced by Langlands in 1967 and shortly thereafter, and recount to some extent their origins. The crucial part of the story took place in January of 1967, when Langlands composed a letter to Weil in which the essential part of his program first saw light. Up to then, Langlands' own work on automorphic forms had certainly been impressive, but that single letter, which cost Langlands a great deal of effort, amounted to a definite turning point. What I have to say in the rest of this paper might be considered a kind of guide to reading both that letter and slightly later material. I will also include some informal remarks of an historical nature, and at the end a somewhat unorthodox collection of unsolved related problems. There are, of course, a number of surveys of this material, notably a few expositions by Langlands himself and that of Borel at the Corvallis conference, but it seems to me that there is still much room left for more of the same.

Incidentally, the letter to Weil was the first document posted on the UBC site.

Roughly speaking, there were two notable features to the letter. The first was that it incorporated in the theory of automorphic forms a radical use of adèle groups and, implicitly, the representation theory of local reductive groups. The second was that it introduced what is now called the L-group. It was the first which attracted a lot of attention—and even perhaps controversy and resentment—at the beginning, but in the long run this was an inevitable step. Furthermore, the incorporation of adèle groups did not originate with Langlands, although in his hands they were to be more important than they had been. But it is the second feature which was really the more significant. In the intervening years, the L-group has come to play a central role in much of the theory of automorphic forms and related fields.

1. Automorphic forms and adèle groups

Classically, the automorphic forms considered in number theory are functions on the Poincaré upper half plane $\mathcal{H}$ satisfying certain transformation properties with respect to a congruence group Γ in $SL_2(\mathbb{Z})$, some partial differential equation involving $SL_2(\mathbb{R})$-invariant differential operators on $\mathcal{H}$, and some growth conditions near cusps. They include, for example, the 'non-analytic' automorphic forms defined first by Maass, which are simply functions on $\Gamma\backslash\mathcal{H}$ and eigenfunctions for the non-Euclidean Laplacian.

Tamagawa tells me it might have been Taniyama who first noticed that one could translate classical automorphic forms to certain functions on adèle quotients. More precisely, let Γ be the principal congruence subgroup of level N in $SL_2(\mathbb{Z})$. Choose a compact open subgroup K_N of $\prod_{p|N} GL_2(\mathbb{Z}_p)$ with these two properties: (1) Γ is the inverse image of K_N with respect to the natural embedding of $SL_2(\mathbb{Z})$ into $\prod_{p|N} GL_2(\mathbb{Z}_p)$; (2) $\det(K_N) = \prod_{p|N} \mathbb{Z}_p^\times$. A common choice is

$$K_N = \left\{ k \;\middle|\; k \equiv \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix} \text{ modulo } N \right\} .$$

For $p \not| N$, let $K_p = GL_2(\mathbb{Z}_p)$. The product $K_f = K_N \prod_{p \not| N} K_p$ is a compact open subgroup of $GL_2(\mathbb{A}_f)$ and Γ is the inverse image of K_f in $SL_2(\mathbb{Q})$. Since $\mathbb{Z}$ is a principal ideal domain, strong approximation tells us that the natural embedding

$$\Gamma\backslash\mathcal{H} \hookrightarrow GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/K_{\mathbb{R}}K_f$$

is a bijection. Here $K_{\mathbb{R}} = SO(2)$, the elements of $GL_2^{\text{pos}}(\mathbb{R})$ fixing i in the usual action on $\mathcal{H}$. Maass' functions on $\Gamma\backslash\mathcal{H}$ may therefore be identified with certain functions on the adèle quotient $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$ fixed by $K_{\mathbb{R}}K_f$, and holomorphic modular forms of weight other than 0 may be identified with functions transforming in a certain way under $K_{\mathbb{R}}$.

If g is any element of $G(\mathbb{A}_f)$ then we can express the double coset $K_f\, g\, K_f$ as a disjoint union of right cosets $g_i\, K_f$, and define the action of a kind of Hecke operator on the space of all functions on $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/K_{\mathbb{R}}K_f$ according to the formula

$$T_g f(x) = \sum f(xg_i) .$$

It is not difficult to see that the Hecke operators T_p and $T_{p,p}$ on $\Gamma\backslash\mathcal{H}$ correspond to right convolution by certain functions on p-adic groups $GL_2(\mathbb{Q}_p)$. More precisely, after a little fussing with weights to deal with the problem that classical Hecke operators involve a left action and the adèlic operators a right one, we can make the classical operators T_p and $T_{p,p}$ for $p \not| N$ correspond to the double cosets

$$K_p \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} K_p, \qquad K_p \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} K_p \,.$$

For classical automorphic forms, where one already has good tools at hand and where terminology is not bad, it might not be entirely clear why this translation to an adèle quotient is a good idea, but in other situations it makes life much simpler immediately. In particular, just as it does elsewhere in number theory, it allows one to separate global questions from local ones. Of course this always makes things clearer, but in this case especially so, and in fact just as with Tate's thesis it raises questions in local analysis which might never have otherwise appeared. For example, already even for classical forms one has to tailor the definition of Hecke operators to the level of the forms involved. In the adèlic scheme, this fiddling takes place in the choice of K_N, and the Hecke operators themselves become entirely local operators (depending only on the prime p).

As one has known since Iwasawa and Tate showed us how to look at ζ-functions, although adèles are a luxury for $\mathbb{Q}$ they are a virtual necessity for other number fields, where problems involving units and class groups, for example, otherwise confuse local and global questions enormously. For the moment, let $\mathbb{A}$ be the adèle ring of F. The exercise above for $GL_2(\mathbb{Q})$ thus suggests the following definition: An **automorphic form** for a reductive group G defined over a number field F is a function on the adèlic quotient $G(F)\backslash G(\mathbb{A})$ with these properties:

- *It satisfies a condition of moderate growth on the adèlic analogue of Siegel sets;*
- *it is smooth at the real primes, and contained in a finite-dimensional space invariant under $Z(\mathfrak{g})$, the centre of the universal enveloping algebra of $G_\mathbb{R}$, as well as $K_\mathbb{R}$, a maximal compact subgroup of $G(\mathbb{R})$;*
- *it is fixed with respect to the right action of some open subgroup K_f of the finite adèle group $G(\mathbb{A}_f)$.*

Hecke operators are determined through convolution by functions on $K_f\backslash G(\mathbb{A}_f)/K_f$.

The conditions on $G_\mathbb{R}$ determine an ideal I of finite codimension in $Z(\mathfrak{g})Z(\mathfrak{k})$, that of differential operators annihiliating the form. One of the fundamental theorems in the subject is that for a fixed I, K_f, and $K_\mathbb{R}$ the dimension of automorphic forms annihilated by I is finite.

The group G will be **unramified** outside a finite set of primes $\mathcal{D}_G$, that is to say arises by base extension from a smooth reductive group over $\mathfrak{o}_F[1/N]$ for some positive integer N. For primes $\mathfrak{p}$ not dividing N, the group $G/F_\mathfrak{p}$ will therefore arise by base extension from a smooth reductive group scheme over $\mathfrak{o}_\mathfrak{p}$. One can express compact open subgroups K_f as a product $K_S \prod_{p \notin S} K_\mathfrak{p}$, where S is a set of primes including $\mathcal{D}_G$ and for $p \notin S$ we have $K_\mathfrak{p} = G(\mathfrak{o}_\mathfrak{p})$. The Hecke operators for K_f will include those defined by double cosets $K_\mathfrak{p}\backslash G(F_\mathfrak{p})/K_\mathfrak{p}$ for $\mathfrak{p}$ not in S.

In one of the next sections I will recall why the algebra generated by the characteristic functions of these cosets is a commutative ring, the local Hecke algebra $\mathfrak{H}_\mathfrak{p}$, whose structure one understands well. In this section I point out only that because of commutativity together with finite-dimensionality, it makes sense—and does no harm here—to impose on an automorphic form the condition that it be an eigenfunction for all but a finite number of Hecke algebras $\mathfrak{H}_\mathfrak{p}$.

From now on, let $\mathcal{A}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$ be the space of automorphic forms on $G(\mathbb{Q})\backslash G(\mathbb{A})$.

2. The constant term of Maass' Eisenstein series

I will illustrate the convenience of adèle groups by calculating in two ways the constant term of Maass' Eisenstein series. In addition to illustrating adèlic techniques, the calculation will be required later on.

Let $\Gamma = SL_2(\mathbb{Z})$. For any complex number s with $\mathrm{REAL}(s) > 1/2$ and any $z = x + iy$ in $\mathfrak{H}$ define

$$E_s(z) = \sum_{\substack{c\ge 0\\ \gcd(c,d)=1}} \frac{y^{s+1/2}}{|cz+d|^{2s+1}}\,.$$

The point of this series is that for

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we have

$$\mathrm{IMAG}(g(z)) = \mathrm{IMAG}\left(\frac{az+b}{cz+d}\right) = \frac{(ad-bc)}{|cz+d|^2}\,\mathrm{IMAG}(z)$$

so that we actually looking at

$$\sum_{\Gamma_P\backslash\Gamma} \mathrm{IMAG}(\gamma(z))^{s+1/2}$$

where Γ_P is the stabilizer of $i\infty$ in Γ. It is generated by integral translations and the scalar matrices $\pm I$, so the function $\mathrm{IMAG}(()z)$ is Γ_P-invariant. The series converges and defines a real analytic function on $\Gamma\backslash\mathfrak{H}$ invariant under Γ such that

$$\Delta E_s = (s^2 - 1/4)E_s\,,$$

where Δ is the non-Euclidean Laplacian. Simple spectral analysis will show that for $\mathrm{REAL}(s) > 1/2$ the function E_s is the unique eigenfunction of Δ on $\Gamma\backslash\mathfrak{H}$ asymptotic to $y^{s+1/2}$ at ∞ in the weak sense that the difference is square-integrable. A little more work will then show that it continues meromorphically in s and is asymptotic to a function of the form

$$y^{1/2+s} + c(s)y^{1/2-s}$$

as $y \to \infty$ for generic s, in the strong sense that the difference between E_s and this asymptotic term is rapidly decreasing in y. (My current favourite explanation of the general theory is the lucid article by Jacquet at the Edinburgh conference, but of course in the particular case at hand one can follow the more elementary technique of Maass.) Of course the coefficient $c(s)$ is a meromorphic function of s. It is easy to deduce that E_s must therefore also satisfy the **functional equation**

$$E_s = c(s)E_{-s}\,,$$

which implies that $c(s)$ satisfies its own functional equation

$$c(s)c(1-s) = 1\,.$$

It turns out also that $y^{1/2+s} + c(s)y^{1/2-s}$ is the **constant term** in the Fourier series of E_s at ∞, which is to say that

$$y^{1/2+s} + c(s)y^{1/2-s} = \int_0^1 E_s(x+iy)\,dx\,.$$

In this section we will calculate $c(s)$ explicitly in both classical and adèlic terms, to get a feel for the way things go in each case.

- *The classical calculation*

The constant term of E_s is

$$
\begin{aligned}
\int_0^1 E_s(x+iy)\,dx &= y^{s+1/2} + y^{s+1/2} \sum_{\substack{c>0 \\ \gcd(c,d)=1}} \int_0^1 \frac{dx}{|cx+icy+d|^{2s+1}} \\
&= y^{s+1/2} + y^{s+1/2} \sum_{\substack{c>0 \\ \gcd(c,d)=1}} \int_0^1 \frac{dx}{|(cx+d)^2+c^2y^2|^{s+1/2}} \\
&= y^{s+1/2} + y^{s+1/2} \sum_{\substack{c>0 \\ \gcd(c,d)=1}} \frac{1}{c^{2s+1}} \int_0^1 \frac{dx}{|(x+d/c)^2+y^2|^{s+1/2}} \\
&= y^{s+1/2} + y^{s+1/2} \left(\int_{-\infty}^{\infty} \frac{dw}{|w^2+y^2|^{s+1/2}} \right) \sum_{c>0} \frac{\varphi(c)}{c^{2s+1}} \quad (w = x + c/d) \\
&= y^{s+1/2} + y^{1/2-s} \left(\int_{-\infty}^{\infty} \frac{dw}{|w^2+1|^{s+1/2}} \right) \sum_{c>0} \frac{\varphi(c)}{c^{2s+1}} \\
&= y^{s+1/2} + y^{1/2-s} \frac{\Gamma(1/2)\Gamma(s)}{\Gamma(s+1/2)} \sum_{c>0} \frac{\varphi(c)}{c^{2s+1}} \\
&= y^{s+1/2} + y^{1/2-s} \frac{\zeta_{\mathbb{R}}(2s)}{\zeta_{\mathbb{R}}(2s+1)} \sum_{c>0} \frac{\varphi(c)}{c^{2s+1}}
\end{aligned}
$$

where

$$
\zeta_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)\,.
$$

Here $\varphi(c)$ is the number of integers mod c relatively prime to c. The terms in the sum are 'multiplicative' in the arithmetic sense, so the sum is also equal to

$$
\begin{aligned}
\prod_p \sum_{n\ge 0} \frac{\varphi(p^n)}{p^{n(2s+1)}} &= \prod_p \left(1 + \sum_{n>0} \frac{(p^n - p^{n-1})}{p^{n(2s+1)}} \right) \\
&= \prod_p \left(1 + (1-1/p) \sum_{n>0} \frac{1}{p^{2ns}} \right) \\
&= \prod_p \left(1 + p^{-2s}(1-1/p) \sum_{n\ge 0} \frac{1}{p^{2ns}} \right) \\
&= \prod_p \left(1 + p^{-2s}(1-1/p) \frac{1}{1-p^{-2s}} \right) \\
&= \prod_p \left(\frac{1 - p^{-2s} + p^{-2s} - p^{-2s-1}}{1-p^{-2s}} \right) \\
&= \prod_p \left(\frac{1-p^{-2s-1}}{1-p^{-2s}} \right)
\end{aligned}
$$

so that

$$c(s) = \frac{\xi(2s)}{\xi(2s+1)}$$

where

$$\xi(s) = \zeta_{\mathbb{R}}(s) \prod_p \frac{1}{1 - p^{-s}} \,.$$

- *The adèlic calculation*

Let $G = SL_2$, and continue to let $\Gamma = SL_2(\mathbb{Z})$. For each finite prime p let $K_p = G(\mathbb{Z}_p)$, and let $K_f = \prod K_p$.

By strong approximation the natural inclusion of $G(\mathbb{R})$ into $G(\mathbb{A})$ induces a bijection

$$\Gamma \backslash \mathfrak{H} \hookrightarrow G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_{\mathbb{R}} K_f$$

or in other words $G(\mathbb{A}) = G(\mathbb{Q}) G(\mathbb{R}) K_f$.

To a function $f(g)$ on $\Gamma \backslash G(\mathbb{R})$ therefore corresponds a unique function F on $G(\mathbb{Q}) \backslash G(\mathbb{A})$ defined by the formula

$$F(g_0\, g_{\mathbb{R}}\, g_f) = f(g_{\mathbb{R}})$$

if g_0 lies in $G(\mathbb{Q})$, $g_{\mathbb{R}}$ in $G(\mathbb{R})$, g_f in K_f. Let $\mathcal{E}_s$ be the function on the adèle quotient corresponding to E_s.

We can in fact define $\mathcal{E}_s$ directly, by imitating the series construction of Eisenstein series on the adèle group.

Before I do this, let me explain simple generalizations of the classical Eisenstein series and constant term. Let P be the subgroup of $G = SL_2$ of upper triangular matrices, N its unipotent radical. Let φ be a function on $\Gamma_P N(\mathbb{R}) \backslash G(\mathbb{R})$ which is a finite sum of eigenfunctions with respect to $SO(2)$. Suppose also that φ satisfies the equation

$$\varphi(pg) = \delta_P^{s+1/2}(p) \varphi(g)$$

where

$$\delta_P \colon \begin{bmatrix} a & x \\ 0 & a^{-1} \end{bmatrix} \mapsto |a|^2$$

is the modulus character of the group P. Then the series

$$\sum_{\Gamma_P \backslash \Gamma} \varphi(\gamma g)$$

will converge to an automorphic form on $\Gamma \backslash G(\mathbb{R})$ if $\mathrm{REAL}(s) > 1/2$, and continue meromorphically in s. If φ is invariant on the right by $SO(2)$ it will be, up to a scalar multiple, the Eisenstein series E_s.

If Φ is an automorphic form on $\Gamma \backslash G(\mathbb{R})$, define its **constant term** to be the function

$$\int_{N(\mathbb{Z}) \backslash N(\mathbb{R})} \Phi(ng)\, dn$$

on $\Gamma_P N(\mathbb{R}) \backslash G(\mathbb{R})$, where $N(\mathbb{Z}) \backslash N(\mathbb{R})$ is assigned measure 1.

Thus if we apply the constant term to an Eisenstein series we get a map from certain functions on $\Gamma_P N(\mathbb{R})\backslash G(\mathbb{R})$ to other functions on the same space. Rather than analyze this in detail, I will now explain what happens for adèle groups.

Because $\mathbb{Q}$ has class number one

$$\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}^\times \prod_p \mathbb{Z}_p^\times \, .$$

Also because $G = PK$ locally we have

$$G(\mathbb{A}) = P(\mathbb{A})\, K_f$$

and hence

$$G(\mathbb{A}) = P(\mathbb{Q}) N(\mathbb{A})\, A(\mathbb{R})\, K_f$$

where A is the group of diagonal matrices in G, or equivalently

$$P(\mathbb{Q})N(\mathbb{A})\backslash G(\mathbb{A})/K_f \cong \Gamma_P N(\mathbb{R})\backslash G(\mathbb{R})\, .$$

Let δ_P be the modulus character of $P(\mathbb{A})$, taking h to the product of all the local factors $\delta_P(h_p)$. Let φ_s be the unique function on $P(\mathbb{Q})N(\mathbb{A})\backslash G(\mathbb{A})/K_\mathbb{R} K_f$ such that $\varphi_s(pg) = \delta_P^{s+1/2}(p)\varphi_s(g)$ and $\varphi(1) = 1$.

- *The function $\mathcal{E}_s$ is the meromorphic continuation of the series*

$$\sum_{P(\mathbb{Q})\backslash G(\mathbb{Q})} \varphi_s(\gamma g)\, .$$

If Φ is an automorphic form on $G(\mathbb{Q})\backslash G(\mathbb{A})$, its **constant term** is defined to be the function

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \Phi(ng)\, dn$$

on $P(\mathbb{Q})N(\mathbb{A})\backslash G(\mathbb{A})$. This is compatible with the classical one in that this diagram is commutative ($\mathcal{A}$ denotes automorphic forms):

$$\begin{array}{ccc} \mathcal{A}(\Gamma\backslash G(\mathbb{R})) & \xrightarrow{f \mapsto \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} f(n)\, dn} & \mathcal{A}(\Gamma_P N_P(\mathbb{R})\backslash G) \\ \downarrow & & \downarrow \\ \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) & \xrightarrow{F \mapsto \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} F(n)\, dn} & \mathcal{A}(P(\mathbb{Q})N_P(\mathbb{A})\backslash G(\mathbb{A})) \end{array}$$

Therefore we calculate the constant term of $\mathcal{E}_s$ to be the function

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \mathcal{E}_s(ng)\, dn = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{P(\mathbb{Q})\backslash G(\mathbb{Q})} \varphi_s(\gamma n g)\, dn\, .$$

The point now is that we can apply the Bruhat decomposition

$$G(\mathbb{Q}) = P(\mathbb{Q}) \cup P(\mathbb{Q}) w^{-1} N(\mathbb{Q}), \quad P(\mathbb{Q})\backslash G(\mathbb{Q}) = \{1\} \cup w^{-1} N(\mathbb{Q})$$

where

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

We can therefore express the constant term as

$$\varphi_s(g) + \int_{N(\mathbb{A})} \varphi_s(w^{-1} n g)\, dn \,.$$

The integral over $N(\mathbb{A})$ is just the product of integrals over all the local groups $N(\mathbb{Q}_p)$. We must therefore calculate the integrals

$$\int_{N(\mathbb{Q}_p)} \varphi_{s,p}(w^{-1} n)\, dn$$

with $\varphi_{s,p}(hk) = \delta_P^{s+1/2}(h)$ $(h \in P(\mathbb{Q}_p))$.

In both real and p-adic cases we start with

$$w^{-1} n = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -x \end{bmatrix} .$$

We now must factor this as hk. In all cases we rely on the transitivity of the action of K on $\mathbb{P}^1(\mathbb{Q}_p)$. The group P is the stabilizer of the image in $\mathbb{P}^1$ of the image of the row vector $[\,0 \quad 1\,]$, so in order to factor $w^{-1}n = pk$ we must find k in K_p taking $[\,0 \quad 1\,]$ to $[\,-1 \quad -x\,]$.

- *The p-adic case*

Let $K = G(\mathbb{Z}_p)$. If x lies in $\mathbb{Z}_p$ then $w^{-1}n$ lies also in K, and there is nothing to be done. Else $|x| = p^{-n}$ with $n > 0$ and $1/x$ lies in $\mathbb{Z}_p$. The row vector $[\,-1 \quad -x\,]$ is projectively equivalent to $[\,x^{-1} \quad 1\,]$ so we may let

$$k = \begin{bmatrix} 1 & 0 \\ x^{-1} & 1 \end{bmatrix} .$$

which gives us

$$h = \begin{bmatrix} 0 & 1 \\ -1 & -x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -x^{-1} & 1 \end{bmatrix} = \begin{bmatrix} -x^{-1} & 1 \\ 0 & -x \end{bmatrix} .$$

For every integer n let

$$N_n = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \,\middle|\, x \equiv 0 \ (\mathrm{mod}\ p^n) \right\} .$$

Calculate

$$\begin{aligned} \int_N f(w^{-1}n)\, dn &= \int_{N_0} f(w^{-1}n)\, dn + \sum_{n<0} \int_{N_n - N_{n+1}} f(w^{-1}n)\, dn \\ &= 1 + (p-1)p^{-(2s+1)} + (p^2 - p)p^{-2(2s+1)} + \cdots \\ &= \frac{1 - p^{-1-2s}}{1 - p^{-2s}} \,. \end{aligned}$$

- *The real group*

Here we normalize $[\,-1 \quad -x\,]$ to $[\,(x^2+1)^{-1/2} \quad x(x^2+1)^{-1/2}\,]$ and let

$$k = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

where

$$\begin{aligned} c &= \frac{x}{\sqrt{x^2+1}} \\ s &= \frac{1}{\sqrt{x^2+1}}\,. \end{aligned}$$

Then

$$h = \begin{bmatrix} 0 & 1 \\ -1 & -x \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} -(x^2+1)^{-1/2} & x(x^2+1)^{-1/2} \\ 0 & -(x^2+1)^{1/2} \end{bmatrix}$$

The integral is therefore

$$\int_{-\infty}^{\infty} (x^2+1)^{-s-1/2}\,dx = \frac{\Gamma(1/2)\,\Gamma(s)}{\Gamma(s+1/2)}\,.$$

These local calculations lead to exactly the same formula for $c(s)$ as before, of course, but it seems fair to claim that we understand better why it has an Euler product.

3. The Satake isomorphism

In this section and the next two I shall explain how the L-group is constructed. Suppose briefly that F is an algebraic number field, $\mathbb{A}$ its adèle ring. Recall that if φ is an automorphic form on $G(F)\backslash G(\mathbb{A})$ then φ is fixed by almost all the local compact groups $G(F_\mathfrak{p})$. We know also from Hecke's analysis of classical automorphic forms that it's important to understand how certain $\mathfrak{p}$-adic **Hecke operators** act on φ.

For almost all primes $\mathfrak{p}$, the local group $G(F_\mathfrak{p})$ is unramified in the sense that G splits over an unramified extension of $F_\mathfrak{p}$, which also means that G can be obtained by base extension from a smooth reductive group scheme over $\mathfrak{o} = \mathfrak{o}_\mathfrak{p}$, the ring of integers of F. The **Hecke algebra** $\mathfrak{H} = \mathfrak{H}(G(F_\mathfrak{p}), G(\mathfrak{o}))$ is defined to be the algebra of measures of compact support on G both right- and left-invariant under the maximal compact subgroup $K = G(\mathfrak{o})$, with convolution as multiplication. It is of course generated by the measures constant on single double cosets with respect to K. Note that we can identify such measures with right K-invariant functions if we are given a Haar measure on G.

From now on in this section, let F be a $\mathfrak{p}$-adic field.

We are interested in homomorphisms of the Hecke algebra $\mathfrak{H}(G(F), G(\mathfrak{o}))$ into $\mathbb{C}$, and more generally in the structure of this algebra.

There is one simple way to obtain such homomorphisms. Let B be a Borel subgroup obtained by base extension from a Borel subgroup of $G(\mathfrak{o})$. Let $\delta = \delta_B$ be the **modulus character** of B, taking b to $|\det_\mathfrak{b}(b)|$. We have an Iwasawa decomposition $G = BK$. Therefore, if χ is a character of B trivial on $B \cap K = B(\mathfrak{o})$ (which is to say an **unramified character** of B), there is a unique function $\varphi = \varphi_\chi$ on G such that

$$\varphi_\chi(bk) = \chi(b)\delta^{1/2}(b)$$

for all b in B, k in K. Up to a scalar multiple, it is unique with the property

$$\varphi(bgk) = \chi(b)\delta^{1/2}(b)\varphi(g)\,.$$

Right convolution by elements of the Hecke algebra $\mathfrak{H}$ preserves this property, hence elements of $\mathfrak{H}$ act simply as multiplication by scalars, and therefore from each χ we obtain a homomorphism Φ_χ from $\mathfrak{H}$ to $\mathbb{C}$.

The normalizing factor $\delta^{1/2}$ is there for several reasons, but among others to make notation easier in the result I am about to mention.

Suppose T to be a maximal torus in B, and w to be an element of K in the associated Weyl group W, and N the unipotent radical of B. If χ satisfies some simple inequalities then the integral

$$\tau_w\varphi_\chi(g) = \int_{wNw^{-1}\cap N\backslash N} \varphi_\chi(w^{-1}ng)\,dn$$

will converge and satisfy the condition

$$\tau_w\varphi_\chi(bk) = w\chi(b)\delta_B^{1/2}(b)\tau_w\varphi_\chi(1)\,.$$

The operator $\varphi_\chi \mapsto \tau_w\varphi_\chi$ also commutes with the Hecke algebra. The function $\tau_w\varphi_\chi$ will therefore be a (generically non-zero) multiple of $\varphi_{w\chi}$. As a consequence, the homomorphisms Φ_χ and $\Phi_{w\chi}$ are the same.

This means that the map $\chi \mapsto \Phi_\chi$ induces one from the W-orbits of unramified characters of T to a set of homomorphisms from the Hecke algebra $\mathfrak{H}$ to $\mathbb{C}$.

There is another way to set this up. Let T be a maximal torus in B and A a maximal split torus in T. The injection of A into T induces an isomorphism of free groups $\mathfrak{A} = A(F)/A(\mathfrak{o})$ with $T(F)/T(\mathfrak{o})$. Restriction from B to A therefore induces an isomorphism of the group of unramified characters of B (or T) with those of A. Let $\mathfrak{R}$ be the group ring $\mathbb{C}[\mathfrak{A}]$ of $\mathfrak{A}$. Because $G = BK$, the $\mathfrak{R}$-module of all K-invariant functions on G with values in $\mathfrak{R}$ such that $f(ntg) = t\,\delta_B^{1/2}(t)f(g)$ is free of rank one over $\mathfrak{R}$. Convolution by elements of the Hecke algebra are $\mathfrak{R}$-homomorphisms of this module, and therefore we have a ring homomorphism Φ from $\mathfrak{H}$ to $\mathfrak{R}$. Any unramified character χ of T gives rise to a ring homomorphism from $\mathfrak{R}$ to $\mathbb{C}$, and the composition of this with Φ will be the same as Φ_χ. The W-invariance we saw before now implies that the image of Φ lies in $\mathfrak{R}^W$. The following result is due in special cases to different people, but put in essentially definitive form by Satake:

Theorem. *The canonical map constructed above from the Hecke algebra $\mathfrak{H}$ to $\mathbb{C}[\mathfrak{A}]^W$ is a ring isomorphism.*

In other words, all homomorphisms from $\mathfrak{H}$ to $\mathbb{C}$ are of the form Φ_χ for some χ. The point of Satake's proof is injectivity.

For example, let G be $GL_2(\mathbb{Q}_p)$, A the group of diagonal matrices in G. Suppose the character χ takes the matrix ϖ_1 with diagonal $(p, 1)$ to α_1 and the matrix ϖ_2 with diagonal $(1, p)$ to α_2. The ring $\mathbb{C}[\mathfrak{A}]^W$ is generated by the images of $\varpi_1 + \varpi_2$ and $(\varpi_1\varpi_2)^{\pm 1}$. The Hecke operator T_p acts on φ_χ through multiplication by $p^{1/2}(\alpha_1 + \alpha_2)$, and $T_{p,p}$ by $\alpha_1\alpha_2$.

4. The dual group I. The split case

Suppose we are given a classical automorphic form of weight k for the congruence group Γ, say an eigenform for the Hecke operator T_p with eigenvalue c_p. Then by Deligne's version of the Weil conjectures $c = (a_p + b_p)$ with $|a_p| = |b_p| = p^{(k-1)/2}$, and also $a_p b_p = p^{k-1}$. The diagonal matrix

$$\begin{bmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{bmatrix}$$

where $\alpha_p = a_p/p^{(k-1)/2}$, $\beta_p = b_p/p^{(k-1)/2}$, is therefore unitary. Of course the pair (α_p, β_p) is determined only up to a permutation. It was remarked by Sato and Tate in a slightly different context that the statistical distribution of the numbers c_p as p varied seemed to be according to the $SU(2)$-invariant measure on the conjugacy classes determined by the pairs (α_p, β_p). This suggests that, more generally, an eigenfunction with respect to Hecke operators ought to be thought of as determining a conjugacy class in a complex group.

The simplest version of Langlands' construction of his dual group does exactly this. But there is a slight twist in the story.

Let G be any split reductive group defined over a $\mathfrak{p}$-adic field F. As before, let $T \subseteq B$ be a maximal split torus contained in the Borel subgroup B, and let W be the Weyl group of this pair (G, T). An eigenfunction with respect to the Hecke algebra of $G(F)$ with respect to the maximal compact subgroup $G(\mathfrak{o})$, according to Satake's theorem, determines an element in the W-orbit of $\mathrm{Hom}(T(F)/T(\mathfrak{o}), \mathbb{C}^\times)$. Following the suggestion of Sato-Tate, we want to interpret this first as a W-orbit in a complex torus, then as a conjugacy class in some reductive group containing that torus.

Let $\widehat{T}$ be the torus we are looking for. We first pose an identification

$$\widehat{T}(\mathbb{C}) = \mathrm{Hom}(T(F)/T(\mathfrak{o}), \mathbb{C}^\times)\,.$$

which means that points on the torus $\widehat{T}$ are the same as unramified characters of $T(F)$. Second, we fix a map

$$\lambda\colon\ T(F)/T(\mathfrak{o}) \longrightarrow X_*(T) = \mathrm{Hom}(X^*(T), \mathbb{Z})$$

which identifies $T(F)/T(\mathfrak{o})$ with the lattice $X_*(T)$ of coweights of T. This map is characterized by the formula

$$|\chi(t)| = q_F^{\langle \chi, \lambda(t)\rangle}$$

for all t in T, χ in $X^*(T)$. Equivalently, if μ is a coweight of T then it is the image of $\mu(\varpi^{-1})$ if ϖ is a generator of $\mathfrak{p}$. This allows to make the identification

$$\widehat{T}(\mathbb{C}) = \mathrm{Hom}(X_*(T), \mathbb{C}^\times)\,.$$

Now if S is any complex torus then we have a canonical identification

$$S(\mathbb{C}) = \mathrm{Hom}(X^*(S), \mathbb{C}^\times)$$

since the coupling

$$S(\mathbb{C}) \times X^*(S) \longrightarrow \mathbb{C}^\times$$

is certainly nondegenerate. If we set $S = \widehat{T}$ we get an identification

$$\widehat{T}(\mathbb{C}) = \mathrm{Hom}(X^*(\widehat{T}), \mathbb{C}^\times)$$

which leads us also to pose

$$X^*(\widehat{T}) = X_*(T)\,.$$

In other words, in some sense the tori T and $\widehat{T}$ must be dual to each another. In any event, this is a natural way to construct tori, since from almost any standpoint a torus is completely determined by its lattice of characters.

In summary:

- *Points on the torus $\widehat{T}(\mathbb{C})$ may be identified with unramified complex characters of $T(F)$.*
- *Elements of $T(F)/T(\mathfrak{o})$ may be identified with rational characters of $\widehat{T}$.*

This kind of duality can be extended to one of reductive groups. Let $\Sigma \subseteq X^*(T)$ be the set of roots of $\mathfrak{g}$ with respect to T, and let $\Sigma^\vee$ be the associated set of coroots in $X_*(T)$. The quadruple $(X^*(T), \Sigma, X_*(T), \Sigma^\vee)$ all together make up the **root data** of the pair (G, T). If, conversely, one is given a quadruple (L^*, S^*, L_*, S_*) where L_* is a free abelian group of finite rank, L^* is the dual of L_*, S^* is a root system in L^* and S_* a compatible coroot system in L_* then we can find a reductive group defined and split over any field with these as associated root data. It is unique up to inner automorphism.

In our case, given the root data $(X^*(T), \Sigma, X_*(T), \Sigma^\vee)$ we get another set of root data by duality, namely the quadruple

$$(X_*(T), \Sigma^\vee, X^*(T), \Sigma) = (X^*(\widehat{T}), \Sigma^\vee, X_*(\widehat{T}), \Sigma)\,.$$

Let $\widehat{G}$ be the reductive group defined over $\mathbb{C}$ associated to these data. For example, if G is simply connected and of type C_n then $\widehat{G}$ is adjoint and of type B_n. It is this involution of types that is at first a bit puzzling.

If we are given a system of positive roots in G, then the corresponding coroots determine also a positive system of roots in $\widehat{G}$, or in other words a Borel subgroup.

Here is Langlands' version of the Satake isomorphism in these circumstances:

Theorem. *For a split group G over a $\mathfrak{p}$-adic field there is a natural bijection between ring homomorphisms from the Hecke algebra to $\mathbb{C}$ and W-orbits in $\widehat{T}(\mathbb{C})$ or equivalently semi-simple $\widehat{G}(\mathbb{C})$-conjugacy classes in $\widehat{T}(\mathbb{C})$.*

Or in yet another form:

Theorem. *For a split group G over a $\mathfrak{p}$-adic field there is a natural ring isomorphism of the Hecke algebra $\mathfrak{H}$ with the representation ring $\mathrm{Rep}(\widehat{G})$.*

Example. For $G = GL_n$ this is all straightforward. An unramified character of the torus of diagonal elements is of the form

$$\begin{bmatrix} x_1 & 0 & & 0 \\ 0 & x_2 & & 0 \\ & & \dots & \\ 0 & & & x_n \end{bmatrix} \mapsto |x_1|^{s_1} \dots |x_n|^{s_n} = q^{m_1 s_1 + \cdots m_n s_n} = t_1^{m_1} \dots t_n^{m_n}$$

if $|x_i| = q^{m_i}$ and $t_i = q^{s_i}$. In fact the character is determined by the array of complex numbers $t = (t_1, \dots, t_n)$. Permuted arrays will give rise to the same Hecke algebra homomorphism. But this means precisely that what really matters is the conjugacy class of the matrix

$$\mathrm{diag}(t_i) = \begin{bmatrix} t_1 & 0 & & 0 \\ 0 & t_2 & & 0 \\ & & \dots & \\ 0 & & & t_n \end{bmatrix}$$

in $GL_n(\mathbb{C})$. This is compatible with what we have said just above, because the dual group of GL_n is again just GL_n.

Example. For SL_n the dual group is $PGL_n(\mathbb{C})$. The torus $\widehat{T}$ is the quotient of the diagonal matrices by the scalars. This can be identified with the group of complex characters of the group of diagonal matrices in SL_n simply by restriction of the corresponding identification for GL_n. In particular, for $n = 2$ the diagonal matrix

$$\begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}$$

corresponds to the character

$$\begin{bmatrix} \varpi^{-1} & 0 \\ 0 & \varpi \end{bmatrix} \mapsto t_1/t_2 \, .$$

Let me explain briefly what the dual group means for automorphic forms. Suppose now that G is a split reductive group defined over a number field F. Let φ be an automorphic form on $G(F)\backslash G(\mathbb{A})$ which is an eigenfunction for the Hecke algebras $\mathfrak{H}_\mathfrak{p}$ for $\mathfrak{p}$ not in a finite set of primes S. This means that for every f in a local Hecke algebra $\mathfrak{H}_\mathfrak{p}$ there exists a constant c_f such that $R_f \varphi = c_f \varphi$. Then for each $\mathfrak{p}$ not in S there exists a unique conjugacy class $\Phi_\mathfrak{p}$ in $\widehat{G}$ with the property that

$$\mathrm{trace}_\pi(\Phi_\mathfrak{p}) = c_f$$

whenever f is an element of the Hecke algebra $\mathfrak{H}_\mathfrak{p}$ and $\pi = \pi_f$ is the corresponding virtual representation of $\widehat{G}$.

The connection between homomorphisms of the local Hecke algebra and conjugacy classes in $\widehat{G}$ is rather straightforward. It may have been noticed by several mathematicians before Langlands called attention to it, but I can find no record of the observation. It is quite likely that, if it had been observed, it simply wasn't felt to be of great enough importance to be worth making explicit. In Langlands' hands, however, the dual group was to serve as an uncanny guide to understanding an enormously wide range of phenomena involving automorphic forms.

5. The dual group II. The unramified case

The first strong hint that the dual group had nearly magical properties arose in Langlands' construction of the analogue of the dual group for arbitrary unramified $\mathfrak{p}$-adic groups. This, too, can be found in the original letter to Weil.

Now suppose only that G is an unramified reductive group defined over the $\mathfrak{p}$-adic field F. I recall that this means it is determined by base extension from a smooth reductive group scheme over $\mathfrak{o}_\mathfrak{p}$.

Let B be a Borel subgroup and T a maximal torus in B, containing a maximal split torus A. Let W be the restricted Weyl group. Satake's theorem asserts that homomorphisms from the Hecke algebra $\mathfrak{H}_\mathfrak{p}$ to $\mathbb{C}$ correspond naturally to W-orbits in $\widehat{A}(\mathbb{C}) = \mathrm{Hom}(A(F)/A(\mathfrak{o}), \mathbb{C}^\times)$. The injection $A \hookrightarrow T$ gives us also an injection $X_*(A) \hookrightarrow X_*(T)$, hence a dual surjection

$$\widehat{T}(\mathbb{C}) \longrightarrow \widehat{A}(\mathbb{C})\,.$$

In these circumstances, when $A \neq T$ it is not at all obvious how conjugacy classes in the dual group relate to W-orbits in $\widehat{A}$. It has always seemed to me that explaining this, although simple enough once seen, was one of Langlands' least obvious and most brilliant ideas. The trick is to incorporate the Galois group in the definition of the dual group.

The group G will split over an unramified extension E/F. Let $\mathcal{G}$ be the Galois group of E/F, $\mathfrak{Frob}$ the Frobenius automorphism. Because G contains a Borel subgroup defined over F, the Galois group permutes the positive roots of G over E, and this gives rise to a homomorphism from $\mathcal{G}$ to the automorphism group of $\widehat{G}$. Langlands defined the full L-group ${}^LG_{E/F}$ to be the semi-direct product $\widehat{G}\rtimes\mathcal{G}$. Here is his remarkable observation:

Langlands' Lemma. *Semi-simple $\widehat{G}(\mathbb{C})$-conjugacy classes in $\widehat{G}(\mathbb{C}) \times \mathfrak{Frob}$ correspond naturally to W-orbits in $\widehat{A}$.*

If g lies in $\widehat{G}$ then

$$g(g_0, \mathfrak{Frob})g^{-1} = (gg_0g^{-\mathfrak{Frob}}, \mathfrak{Frob})$$

so that $\widehat{G}$-conjugacy in $\widehat{G} \times \mathfrak{Frob}$ is the same as **twisted conjugacy**. On the other hand: (1)

$$g_0^{\mathfrak{Frob}} = g_0^{-1} g_0 g^{\mathfrak{Frob}}$$

and (2) if $\sigma = \mathfrak{Frob}^n$ then

$$\mathfrak{Frob}(g_0, \mathfrak{Frob})\mathfrak{Frob}^{-1} = (\mathfrak{Frob} g_0 \mathfrak{Frob}^{-1}, \mathfrak{Frob}) = (g_0^{\mathfrak{Frob}}, \mathfrak{Frob})\,.$$

Therefore $\widehat{G}$-conjugacy in $\widehat{G} \times \mathfrak{Frob}$ is the same as ${}^LG$-conjugacy.

I outline here explicitly how the correspondence goes. First of all, every semi-simple $\widehat{G}(\mathbb{C})$-conjugacy class in $\widehat{G}(\mathbb{C}) \times \mathfrak{Frob}$ contains elements of the form $t \times \mathfrak{Frob}$ with t in $\widehat{T}(\mathbb{C})$. Second, the W-orbit of image of t in $\widehat{A}(\mathbb{C})$ depends only on the original conjugacy class. This at least gives us a map from these conjugacy classes to W-orbits in $\widehat{A}(\mathbb{C})$. Finally, this map is a bijection.

The simplest published proof of this Lemma can be found in Borel's Corvallis lecture. Like Langlands' original proof, it relies upon an old paper of Gantmacher's for a crucial point, but Kottwitz has pointed out to me that this point follows easily from a well known result of Steinberg's. This is explained briefly in a paper by Kottwitz and Shelstad, and I will sketch here without details a proof which combines the arguments of Borel and Kottwitz-Shelstad.

• The restricted Weyl group is defined to be the image in $\mathrm{Aut}(A)$ of subgroup of the full Weyl group of the pair (G, T) which takes A into itself. In §6.1 of Borel's lecture it is shown that in the dual group $\widehat{G}$ the elements of W can be characterized as those elements of $N_{\widehat{G}}(\widehat{T})/\widehat{T}$ commuting with the Frobenius. • Furthermore, §6.2 of Borel shows that every element of W can be represented by an element of $N_{\widehat{G}}(\widehat{T})$ commuting with the Frobenius. • For s in $\widehat{T}(\mathbb{C})$, conjugation of $t \times \mathfrak{Frob}$ by s is equal to $t(s/s^{\mathfrak{Frob}}) \times \mathfrak{Frob}$. The kernel of the projection from $\widehat{T}$ to $\widehat{A}$ is that spanned by elements $s/s^{\mathfrak{Frob}}$. From this it is easy to see (§6.4 of Borel) that the projection from $\widehat{T}$ to $\widehat{A}$ induces a bijection of $(\widehat{T}(\mathbb{C}) \times \mathfrak{Frob})/N_{\widehat{G}}(\widehat{T})$ with $\widehat{A}(\mathbb{C})/W$. • Every semi-simple conjugacy class in $\widehat{G}(\mathbb{C}) \times \mathfrak{Frob}$ contains an element $t \times \mathfrak{Frob}$ with t in $\widehat{T}$. This is where Borel and Langlands quote Gantmacher, but I present here the argument of Kottwitz and Shelstad.

Let $\widehat{B}$ be a Borel subgroup fixed by $\mathfrak{Frob}$ containing $\widehat{T}$. Given a semi-simple element $x \times \mathfrak{Frob}$ in $\widehat{G}(\mathbb{C})$, we want to find g in $\widehat{G}(\mathbb{C})$ such that

$$g(x \times \mathfrak{Frob})g^{-1} = gxg^{-\mathfrak{Frob}} \times \mathfrak{Frob} = t \times \mathfrak{Frob}$$

with t in $\widehat{T}(\mathbb{C})$. Equivalently, we want to find g with the property that if we set

$$t = gxg^{-\mathfrak{Frob}}$$

then

$$t\widehat{B}t^{-1} = \widehat{B}, \quad t\widehat{T}t^{-1} = \widehat{T}\,.$$

Let $H = \widehat{B}$ or $\widehat{T}$. Then $tHt^{-1} = H$ means that

$$gxg^{-\mathfrak{Frob}}Hg^{\mathfrak{Frob}}x^{-1}g^{-1} = H$$

or equivalently

$$\begin{aligned} x(g^{-1}Hg)^{\mathfrak{Frob}}x^{-1} &= (x \times \mathfrak{Frob})g^{-1}Hg(x \times \mathfrak{Frob})^{-1} \\ &= g^{-1}Hg \end{aligned}$$

since $H^{\mathfrak{Frob}} = H$. In other words, since all Borel subgroups and tori are conjugate in $\widehat{G}(\mathbb{C})$ we are looking for a group H_* fixed under conjugation by $x \times \mathfrak{Frob}$. But a well known result of Steinberg guarantees that we can find some pair (B_*, T_*) fixed by conjugation under $x \times \mathfrak{Frob}$, so we are finished.

• The map induced by inclusion from $(\widehat{T}(\mathbb{C}) \times \mathfrak{Frob})/N_{\widehat{G}}(\widehat{T})$ into the $\widehat{G}$-classes in $\widehat{G} \times \mathfrak{Frob}$ is an injection. This is proven in §6.5 of Borel (but note that there are quite a few simple typographical errors there).

This Lemma has as immediate consequence:

Theorem. *There is a natural bijection between homomorphisms from the Hecke algebra $\mathfrak{H}$ into $\mathbb{C}$ and semi-simple $\widehat{G}(\mathbb{C})$ or ${}^LG(\mathbb{C})$-conjugacy classes in $\widehat{G}(\mathbb{C}) \times \mathfrak{Frob}$.*

Example. *The unramified special unitary group SU_3*

Let $F_\bullet$ be an unramified quadratic extension of F. Let

$$w = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and let G be the unitary group of 3×3 matrices X with coefficients in $F_\bullet$ corresponding to the Hermitian matrix w. This is an algebraic group over F—if R is any ring containing F then $G(R)$ is made up of the matrices with coefficients in $F_\bullet \otimes_F R$ such that

$$Xw\,{}^t\overline{X} = w, \quad w\,{}^tX^{-1}w^{-1} = \overline{X}$$

where $x \mapsto \bar{x}$ comes from conjugation in $F_\bullet$.

The group $G(F)$ contains the torus T of diagonal matrices

$$\begin{bmatrix} y & 0 & 0 \\ 0 & \overline{y}/y & 0 \\ 0 & 0 & 1/\overline{y} \end{bmatrix}$$

with y in $F_\bullet^\times$.

Over $F_\bullet$ this group becomes isomorphic to SL_3, so $\widehat{G}$ is $PGL_3(\mathbb{C})$. Let $\mathcal{G} = \{1, \sigma\}$ be the Galois group of $F_\bullet/F$. The torus $\widehat{T}$ dual to T is the quotient of the group of complex matrices

$$\operatorname{diag}(t_i) = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}$$

by the scalar matrices. The element σ acts on $\widehat{G}$ through the automorphism

$$X \mapsto w\,{}^t\!X^{-1} w^{-1}\,.$$

and, more explicitly, it acts on $\widehat{T}$ by taking

$$\begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix} \mapsto \begin{bmatrix} t_3^{-1} & 0 & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & t_1^{-1} \end{bmatrix}.$$

The map

$$\widehat{T} \longrightarrow \operatorname{Hom}(T(E_\bullet)/T(\mathfrak{o}_{E_\bullet}), \mathbb{C}^\times)$$

takes the element $\operatorname{diag}(t_i)$ to the character

$$\begin{bmatrix} \varpi^{-1} & 0 & 0 \\ 0 & \varpi & 0 \\ 0 & 0 & 1 \end{bmatrix} \mapsto t_1/t_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \varpi^{-1} & 0 \\ 0 & 0 & \varpi \end{bmatrix} \mapsto t_2/t_3\,.$$

The group A is generated by the element

$$\begin{bmatrix} \varpi^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varpi \end{bmatrix}$$

and the map from $\widehat{T}$ to $\widehat{A}$ takes $\operatorname{diag}(t_i)$ to the character

$$\begin{bmatrix} \varpi^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varpi \end{bmatrix} \mapsto t_1/t_3\,.$$

If χ is any unramified character of T, or equivalently of A, the element $\hat{t}_\chi$ can thus be chosen as any element of $\widehat{T}$ such that $t_1/t_3 = \chi(\alpha^\vee(\varpi^{-1}))$.

We shall need to know a bit more about the action of the Galois group on $\widehat{G}$. Let $x_{i,j}$ for $i < j$ be the matrix with a single non-zero entry 1 in location (i,j). These form a basis for the Lie algebra $\hat{\mathfrak{n}}$. Then σ interchanges $x_{1,2}$ and $x_{2,3}$ and takes $x_{1,3}$ to $-x_{1,3}$. This concludes my discussion of SU_3.

Globally, an automorphic form is unramified at all but a finite number of of primes. At an unramified prime $\mathfrak{p}$ it gives rise to a homomorphism from the Hecke algebra $\mathfrak{H}_\mathfrak{p}$ into $\mathbb{C}$. It therefore also corresponds to a semi-simple $\widehat{G}(\mathbb{C})$-conjugacy class $\Phi_\mathfrak{p}$ in $\widehat{G}(\mathbb{C}) \times \mathfrak{Frob}_\mathfrak{p}$ for all but a finite set of $\mathfrak{p}$. It is tempting to call this class the **Frobenius class** of the form at $\mathfrak{p}$, and I shall not resist the temptation.

This construction depends very weakly on the choice of splitting extension E/F, and one has a local L-group for every possible choice. In some ways the canonical choice is to let E be the maximal unramified extension of F.

One can also define an L-group attached to a global field F to be a semi-direct product

$$^L G = \widehat{G} \rtimes \mathcal{G}(\overline{F}/F)$$

and then one has also various embeddings of the local groups into this corresponding to local embeddings of Galois groups. Other variants of the L-group are also possible, with the Galois groups replaced by Weil groups or a Weil-Deligne group. It was at any rate the introduction of Galois groups into the L-group which turned out to be incredibly fruitful. Incidentally, note that the definition of the extended L-group given in this section is compatible with that in the previous one, since when G is split the Galois group acts trivially on $\widehat{G}$.

Langlands himself has told me that the subtle point in his definition of the L-group was not the introduction of the Galois group, which he claims was more or less obviously necessary. Instead, he says, the point about which he worried was that the L-group should be a semi-direct product of $\mathcal{G}$ and $\widehat{G}$ rather than some non-trivial extension. I suppose he had the Weil group—a highly non-trivial extension—on the periphery of his mind. By now there is no doubt that his definition is correct, but it would be an interesting exercise to put together a simple argument to this effect.

6. The dual group III. Why is the L-group important?

There were two questions which were answered, at least conjecturally, as soon as the L-group was defined.

- *How do we attach L-functions to automorphic forms?*

In 1967 there had been already a long history of how to associate L-functions to automorphic forms in very special circumstances, but there was no systematic way to do this. In some cases it was not at all clear which was best among several choices. In terms of the L-group there was a natural guess. Suppose that φ is an automorphic form for a reductive group G defined over a number field F. Let $\Phi_\mathfrak{p}$ be a representative of the corresponding Frobenius class of the L-group for $\mathfrak{p}$ outside some finite set S of primes. Then for each irreducible finite-dimensional representation ρ of the L-group we define

$$L(s,\rho,\varphi) = \prod_{\mathfrak{p}\notin S} \det\left(I - \frac{\rho(\Phi_{\mathfrak{p}})}{N\mathfrak{p}^s}\right)^{-1} .$$

Of course there are a finite number of factors missing for primes in S, but these will not affect analytic properties seriously. Of course one conjectures this L-function to have all sorts of nice analytic properties—meromorphic continuation, functional equation, etc. The new feature here is the parameter ρ, and implicit in the construction of these functions was that ρ should play a role here analogous to that played by representations of the Galois group in the context of Artin's conjecture. This conjecture was made somewhat more reasonable, at least in Langlands' own mind and in lectures he gave very shortly after he wrote the letter to Weil, when he showed that the theory of Eisenstein series provided some weak evidence for it. It turned out that the constant term of series associated to cusp forms on maximal parabolic subgroups determined a new class of L-functions of Langlands' form for which one could at least prove analytic continuation. This was explained in Langlands' Yale notes in Euler products, and I shall say something about it further on. Later and more striking evidence that the L-functions suggested by Langlands were the natural ones was provided by several investigations which showed that the Hasse-Weil ζ functions of Shimura varieties were of Langlands' type. A result of this kind had been shown first by Eichler for classical modular varieties and later on by Shimura for more sophisticated modular varieties, but of course the relationship with the L-group was disguised there. What was really striking was that Deligne's formulation of Shimura's results on modular varieties and their canonical fields of definition fitted naturally with Langlands' L-group. This was first pointed out in Langlands' informal paper on Shimura varieties in the Canadian Journal of Mathematics, recently reprinted.

- *How are automorphic forms on different groups related?*

There were many classical results, culminating in work of Eichler and Shimizu, that exhibited a strong relationship between automorphic forms for quaternion division algebras over $\mathbb{Q}$ and ones on $GL_2(\mathbb{Q})$. To Langlands this appeared as a special case of a remarkable principle he called **functoriality**. The functoraliity principle conjectured that if G_1 and G_2 were two rational reductive groups, then whenever one had a group homomorphism from ${}^LG_1$ to ${}^LG_2$ compatible with projections onto the Galois group, one could expect a strong relationship between automorphic forms for the two groups.

The underlying idea here is perhaps even more remarkable. We know that an automorphic form gives rise to Frobenius classes in local L-groups for all but a finite number of primes. We know that L-functions can be attached to automorphic forms in terms of these classes. The **functoriality principle** asserts that the automorphic form is in some sense very strongly determined by those classes, in that the form has incarnations on various groups determined by maps among these classes induced by L-group homomorphisms. Evidence for this idea was the theorem of Jacquet-Langlands in their book on GL_2 which extended the work of Eichler-Shimizu to arbitrary global fields. This theorem was the first of many to come suggested by the functoriality principle, and its proof was the first and simplest of many in which the trace formula was combined with difficult local analysis.

The functoriality principle was especially interesting when the group G was trivial! In this case the L-group is just its Galois group component, and the functoriality principle asserts that finite dimensional representations of the Galois group should give rise to automorphic forms. This is because an n-dimensional representation of the Galois group amounts to a homomorphism from the trivial L-group into that for GL_n. Even more remarkable was the eventual proof by Langlands of certain non-trivial cases of Artin's conjecture, applying techniques from representation theory and automorphic forms. This was also strong evidence of the validity of the functoriality principle.

7. How much of this was in the letter to Weil?

Essentially all of it! At least the results. The proofs were crude or barely sketched, but better was perhaps not possible in view of incomplete technology. For example, even the work of Bruhat-Tits on the structure of local $\mathfrak{p}$-adic groups was not yet in definitive form. I have always found it astonishing that Langlands introduced the L-group full-grown right from the start. The scope and audacity of the conjectures in his letter to Weil were nearly incredible, especially because at that time the details of various technical things he needed hadn't been quite nailed down yet.

The first reasonably complete account appeared in Langlands' lecture in 1970 at a conference in Washington, the written version in the conference proceedings in the Springer Lecture Notes #170. It is instructive for the timid among us to compare this account with the original letter, and with Godement's account in the Séminaire Bourbaki.

8. Where does representation theory enter?

So far I haven't made any explicit reference to the representation theory of local reductive groups. I haven't needed it to formulate results, but without it the whole subject is practically incoherent. It already appears at least implicitly in the classical theory of automorphic forms, where one always knew that different automorphic forms were only trivially different from others. In current terminology this is because they occurred in the same local representation spaces. One place where local representation theory explains what is really going on is in the treatment of unramified automorphic forms above, where homomorphisms of Hecke algebras were attached to unramified characters χ. Many things look rather bizarre unless we realize that we are looking there at the subspace of $G(\mathfrak{o}_\mathfrak{p})$-fixed vectors in the representation of $G(F_\mathfrak{p})$ induced by χ from the Borel subgroup. In fact, we are really defining a class of local L-functions $L(s,\rho,\pi)$ where now π is an unramified representation of a $\mathfrak{p}$-adic group. Satake's isomorphism asserts that there is a natural bijection between certain $\widehat{G}(\mathbb{C})$-conjugacy classes in local L-groups ${}^LG$ and irreducible unramified representations of the local group $G(F)$. We can reformulate this result by saying that, given on G the structure of a smooth reductive group over $\mathfrak{o}_F$ there is a natural bijection between irreducible unramified representations and splittings of a sequence

$$1 \to \widehat{G}(\mathbb{C}) \to {}^LG \to \langle \mathfrak{Frob} \rangle \to 1\,.$$

(using a suitable variant of ${}^LG$).

This is a special case of a **local functoriality principle**, which conjectures a strong relationship between homomorphisms from a local Galois group into ${}^LG$ and irreducible representations of the local group G. We know that at least for unramified representations π of G we have a whole family of L-functions $L(s,\rho,\pi)$ which vary with the finite dimensional representation ρ of $\widehat{G}$. This leads us to ask more generally how we might associate L functions to representations other than the unramified ones. We know from Tate's thesis in the case of $\mathbb{G}_m = GL_1$ that we should expect not only an L-function but in addition a local **root number** $\epsilon(s,\pi,\rho,\psi)$ as well which depends on a choice of local additive character ψ of the field. This idea was worked out in detail by Jacquet and Langlands for the case of GL_2 through the theory of Whittaker models, and a bit later by Godement and Jacquet for all groups GL_n, following Tate and Weil for division algebras. There are in fact several ways to attach both L-functions and root numbers to representations of a group G defined over a local field $F_{\mathfrak{p}}$, but the most natural and intriguing idea is this, which extends local class field theory in a remarkable way:

- *To each representation of a local reductive group G we should be able to associate a homomorphism from the Galois group or some variant (such as the Weil-Deligne group) into ${}^LG$ which is compatible with the canonical projection from ${}^LG$ onto the Galois group. If ρ is a finite-dimensional representation of ${}^LG$ we can then expect the corresponding L-function and root number to be that obtained by Artin, Hasse, Dwork, and Langlands from the representation of the Galois group we get by composition.*

There are a few mild but important modifications of this idea necessary, for example, to deal with certain poorly behaved ℓ-adic Galois representations, but although evidence for it is still somewhat indirect it seems very likely to be true. In particular if $G = GL_n$ we should expect irreducible cuspidal representations of G to correspond bijectively and naturally to irreducible n-dimensional representations of the Galois group. In my opinion the strongest evidence for the conjecture here comes from work of Deligne, Langlands, and Carayol on the reduction of classical modular varieties in bad characteristic. Here $G = GL_2$. Other convincing evidence comes from the remarkable results of Kazhdan and Lusztig dealing with the best of the poorly behaved cases, covered by the Deligne-Langlands conjecture.

9. Weil's reaction

Weil's first reaction to the letter Langlands had written to him was perhaps not quite what Langlands had hoped for. Langlands had written the letter by hand, and Weil apparently decided that the handwriting was unreadable! You can form your own opinion on this question, because at the UBC Sun SITE we have posted a copy of the hand-written letter in digital format (Weil's very own copy of the original was scanned by Mark Goresky in Princeton). At any rate, Langlands then sent to Weil a typed version. Copies of this were distributed to several mathematicians over the next few years, and this is how Langlands' ideas became well known.

I do not believe that Weil ever made a written reply, but after all he worked only across Princeton from Langlands. Nonetheless, I think it is reasonable to guess that his first serious reaction was confusion. In spite of the fact that Weil had been one of the founders of the theory of algebraic groups, he may not have been familiar with the general theory of root systems, and this alone may have caused him technical difficulty. It also seems that although he had had a hand in introducing representations into the theory of automorphic forms through his papers on Siegel's formulas, he was unfamiliar with the representation theory of Gelfand and Harish-Chandra, which was a major part of Langlands' own background. He says himself of his reaction to Langlands' letter (*Collected Papers III*, page 45)

> … *pendant longtemps je n'y compris rien* …

At the time he received the letter, he was concerned with extending his 'converse theorem', which asserted that if an L-function of a certain type, if it and sufficiently many twists satisfied a certain type of functional equation, arose from a classical cusp form. He wanted to generalise this to automorphic forms for the groups GL_2 associated to number fields other than $\mathbb{Q}$, and troubles he was having with complex archimedean primes were almost immediately cleared up by Langlands' ideas about the relation between representation theory and Galois representations. Also, after a while he worked out the case of GL_n in some detail and gave a talk on it at Oberwolfach. In this case, as we have seen, the L-group is just GL_n again, and many technical difficulties vanish. Finally, he wrote a short paper related to the local conjecture for GL_2 over a $\mathfrak{p}$-adic field with residue characteristic two.

Weil also felt strongly, as he repeated often, that conjectures were to be evaluated according to the evidence behind them. There is much to be said for this attitude, since ideas often come cheaply and without support. Since Langlands' conjectures included Artin's conjecture about L-functions as a special case, and since it took a lot of work to verify even simple cases, or at least a lot of imagination to see how fruitful the conjectures would be in breaking up large problems into smaller ones, it could have been predicted that Weil would be skeptical. What he himself says is this (*Collected Papers III*, page 457):

> ... *je fus incapable de partager l'optimisme de Langlands à ce sujet; la suite a prouvé que j'avais tort. Je lui dis cependant, comme j'ai coutume de le faire en pareil cas: "Theorems are proved by those who believe in them."*

Presumably a necessary, not a sufficient, condition.

10. L-functions associated to the constant term of Eisenstein series

Implicit in Langlands' conjectures is the idea that the L-functions he defines are precisely those of arithmetic interest. Not quite a conjecture, this should be taken rather as a working hypothesis. At the time he made the principal conjectures, the main evidence that he had for this idea came from the theory of Eisenstein series. In this section I will explain this evidence, and even a mild extension of what was known definitely to Langlands in 1967. Other explanations of this material can be found in Langlands' notes on *Euler products* and Godement's Bourbaki talk on the same topic. For technical reasons, both restricted themselves to the case of automorphic forms unramified at all primes of a split group. Developments in local representation theory that took place a few years later made made it possible to extend the result somewhat beyond what can be found in the literature.

The basic idea is simple, but unfortunately it requires some technical preparation to introduce it. Let G be a semi-simple group defined over the number field F, P a rational parabolic subgroup with unipotent radical N and reductive quotient M. For the moment, let $\mathbb{A}$ be the adèle ring of F. We can identify the induced representation

$$\mathrm{Ind}\big(\mathcal{A}\big(M(F)\backslash M(\mathbb{A})\big) \bigm| P(\mathbb{A}), G(\mathbb{A})\big)$$

with a space of functions on $P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})$, which we can call without trouble the space $\mathcal{A}\big(P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})\big)$ of automorphic forms on the parabolic quotient $P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})$. The functions in this space can also be characterized directly.

Suppose that (π, V) is an irreducible representation of $G(\mathbb{A})$ occurring in the subspace of induced cusp forms on $P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})$. Let P_i for $i = 1, \ldots, n$ be the maximal proper rational parabolic subgroups containing P, for each i let δ_i be the modulus character of P_i, and for s in $\mathbb{C}^n$ let

$$\delta_P^s = \prod \delta_i^{s_i} \, .$$

For φ in V and s in $\mathbb{C}^n$ the function

$$\varphi_s = \varphi\, \delta_P^s$$

also lies in the space of induced cusp forms. For $\text{REAL}(s)$ sufficiently large the **Eisenstein series**

$$E[\varphi_s](g) = \sum_{P(F)\backslash G(F)} \varphi_s(\gamma g)$$

converges to an automorphic form on $G(F)\backslash G(\mathbb{A})$, and continues meromorphically in s to all of $\mathbb{C}^n$.

If Φ is an automorphic form on $G(F)\backslash G(\mathbb{A})$ and Q is a rational parabolic subgroup, then the **constant term** of Φ associated to Q is the function

$$\int_{N_Q(F)\backslash N_Q(\mathbb{A})} \Phi(ng)\, dn$$

on $Q(F)N_Q(\mathbb{A})\backslash G(\mathbb{A})$.

Suppose now that P and Q are two rational parabolic subgroups. Start with φ in the space of cusp forms in $\mathcal{A}\big(P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})\big)$, and take the constant term of $E[\varphi]$ with respect to Q. In effect, we are constructing a map from a subspace of

$$\mathcal{A}\big(P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})\big) \longrightarrow \mathcal{A}\big(Q(F)N_Q(\mathbb{A})\backslash G(\mathbb{A})\big)\,.$$

Formally, this is simple to describe. We calculate

$$\int_{N_Q(F)\backslash N_Q(\mathbb{A})} E[\varphi](ng)\, dn = \int_{N_Q(F)\backslash N_Q(\mathbb{A})} \sum_{P(F)\backslash G(F)} \varphi(\gamma ng)\, dn\,.$$

Let T be a maximal split torus contained in both P and Q. The Bruhat decomposition tells us that $P(F)\backslash G(F)/Q(F)$ is a finite disjoint union $P(F)wQ(F)$ as w ranges over an easily described subset of the Weyl group of T in G. We can choose representatives of the Weyl group in $G(F)$. Hence we can write

$$G(F) = \bigcup_w P(F)wQ(F), \quad P(F)\backslash G(F) = \bigcup w\,(w^{-1}P(F)w \cap Q(F))\backslash Q(F)\,.$$

The constant term of $E[\varphi]$ is then the sum

$$\sum_w \int_{N_Q(F)\backslash N_Q(\mathbb{A})} \sum_{(w^{-1}P(F)w\cap Q(F))\backslash Q(F)} \varphi(w\gamma ng)\, dn\,.$$

How to manipulate this expression in the most general case is a bit complicated. There is only one case that we are actually interested in, however—that when Q is an opposite $\overline{P}$ of P. In this case, we may identify the reductive group M with the intersection $P \cap \overline{P}$. Furthermore, there is only one term in the sum that we are interested in, that with $w = 1$. The term we are interested in is then

$$\begin{aligned}\tau\varphi(g) &= \tau_{\overline{P},P}\varphi(g) \\ &= \int_{\overline{N}(F)\backslash\overline{N}(\mathbb{A})} \sum_{P(F)\cap\overline{P}(F)\backslash\overline{P}(F)} \varphi(\gamma\bar{n}g)\, d\bar{n} \\ &= \int_{\overline{N}(F)\backslash\overline{N}(\mathbb{A})} \sum_{\overline{N}(F)} \varphi(\gamma\bar{n}g)\, d\bar{n} \\ &= \int_{\overline{N}(\mathbb{A})} \varphi(\bar{n}g)\, d\bar{n}\,.\end{aligned}$$

We know that π may be expressed as a restricted tensor product $\pi = \widehat{\otimes}\pi_{\mathfrak{p}}$ and hence may assume that φ also is a restricted tensor product $\widehat{\otimes}\varphi_{\mathfrak{p}}$. We may therefore express the adèlic integral as a product

$$\int_{\overline{N}(\mathbb{A})} \varphi(\bar{n}g)\, d\bar{n} = \prod_{\mathfrak{p}} \int_{\overline{N}(F_{\mathfrak{p}})} \varphi_{\mathfrak{p}}(\bar{n}_{\mathfrak{p}}g_{\mathfrak{p}})\, d\bar{n}_{\mathfrak{p}}$$

of local intertwining operators. We shall calculate some of these in a moment. But whether they can be calculated explicitly or not it is known that all of them have a meromorphic continuation in s. For the finite primes this follows from a simple algebraic argument about the Jacquet module, while for the real primes it is somewhat more difficult. At any rate, this point now appears relatively straightforward, but in 1967 it was not known, and appeared difficult.

The representations $\pi_{\mathfrak{p}}$ will be unramified at all but a finite number of primes, and as we shall see in a moment in certain cases the constant term of the Eisenstein series can be written as a quotient of Langlands' L-functions for the inducing representation σ of M. The Eisenstein series satisfies a functional equation

$$E[\varphi] = E[\tau\varphi]$$

and from it we shall deduce that at least in favourable circumstances some of Langlands' Euler products possess a meromorphic continuation also. This argument does not allow us to deduce a functional equation for them, although it is is compatible with one. Because of the technical problems with local intertwining operators, Langlands restricted himself in his writings to globally unramified automorphic forms. Presumably in order to simplify the argument for an untutored audience, he also restricted himself to split groups.

The principal step in this discussion is to express the unramified terms in the product through the L-group. I will do this in detail for unramified rank one groups. The general case will follow easily.

For the moment, let F be an arbitrary $\mathfrak{p}$-adic field.

It suffices to look only at simply connected groups of rank one. There are then two types of unramified $\mathfrak{p}$-adic groups to be considered. The first is the restriction to F of a group $SL_2(E)$ where E/F is an unramified extension. The second is the restriction of a unitary group in three variables.

- *The group $SL_2(E)$*

Let q_E be the size of the residue field $\mathfrak{o}_E/\mathfrak{p}_E$, and let n be the degree of the unramified extension E/F. Let P be the group of upper triangular matrices in $SL_2(E)$, $\overline{P}$ that of lower triangular matrices. Let

$$\chi = \chi_s\colon \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \mapsto |x|_E^s$$

be an unramified character of $P(E)$. Let τ be the $G(E)$-covariant map

$$\mathrm{Ind}\big(\chi_s \mid P(E), G(E)\big) \longrightarrow \mathrm{Ind}\big(\chi_s \mid \overline{P}(E), G(E)\big)$$

defined as the meromorphic continuation of

$$\tau\varphi(g) = \int_{\overline{N}(E)} \varphi(\bar{n}g)\, d\bar{n}\ .$$

Let φ_s be the function in $\mathrm{Ind}\big(\chi_s \mid P(E), G(E)\big)$ fixed by $G(\mathfrak{o}_E)$ with $\varphi_s(1) = 1$, $\overline{\varphi}_s$ the analogous function in $\mathrm{Ind}\big(\chi_s \mid \overline{P}(E), G(E)\big)$. We know that

$$\tau\varphi_s = c(s)\overline{\varphi}_s$$

for some scalar $c(s)$. From the calculation we made before for $SL_2(\mathbb{Q}_p)$ we can deduce that

$$\begin{aligned} c(s) &= \frac{1 - q_E^{-1}\chi(\alpha^\vee(\varpi))}{1 - \chi(\alpha^\vee(\varpi))} \\ &= \frac{1 - q_E^{-1-s}}{1 - q_E^{-s}}\ . \\ &= \prod_0^{n-1} \left(\frac{1 - \omega^i q_F^{-1-s}}{1 - \omega^i q_F^{-s}} \right) \end{aligned}$$

where ω is a primitive n-th root of unity, since $q_E = q_F^n$.

On the other hand, the L-group associated to the restriction of SL_2 from E to F is the semi-direct product of the cyclic Galois group $\mathcal{G}$ with the direct product of n copies of $PGL_2(\mathbb{C})$, $\mathcal{G}$ acting by cyclic permutation. Let $\hat{\mathfrak{n}}^{\mathrm{opp}} = \hat{\mathfrak{n}}_{-\alpha^\vee}$ root space of $\hat{\mathfrak{g}}$ corresponding to the dual root $-\alpha^\vee$, $\hat{t}_\chi$ the element of $\widehat{T}$ corresponding to χ_s. We can write the formula for $c(s)$ in the form

$$c(s) = \frac{\det\big(I - \mathrm{Ad}_{\hat{\mathfrak{n}}^{\mathrm{opp}}}(\hat{t}_\chi)\big)^{-1}}{\det\big(I - q_F^{-1}\mathrm{Ad}_{\hat{\mathfrak{n}}^{\mathrm{opp}}}(\hat{t}_\chi)\big)^{-1}}\ .$$

- *The group SU_3*

Continue to let E be an unramified extension of F and $E_\bullet$ an unramified quadratic extension of E. Let

$$w = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and let G be the unitary group associated to the Hermitian matrix w, already introduced earlier in this paper. The upper triangular matrices in G form a Borel subgroup B, and its opposite can be taken to be the lower triangular matrices. The radical $\overline{N}$ of its opposite is the group of elements

$$\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{bmatrix}$$

where

$$x = \overline{y}, \quad z + \overline{z} = xy\,.$$

The groups B and $\overline{B}$ intersect in the torus T of diagonal matrices

$$\begin{bmatrix} y & 0 & 0 \\ 0 & \overline{y}/y & 0 \\ 0 & 0 & 1/\overline{y} \end{bmatrix}$$

with y in $E_\bullet^\times$.

We want to calculate the constant $c(s)$ such that

$$\tau\varphi_s = c(s)\overline{\varphi}_s\,.$$

We have

$$c(s) = \tau\varphi_s(1) = \int_{\overline{N}(F)} \varphi_s(\bar{n})\, dn\,.$$

If

$$\bar{n} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{bmatrix}$$

then $\bar{n}$ will be in $G(\mathfrak{o})$ if and only if z lies in $\mathfrak{o}_{E_\bullet}$. Otherwise we want to write it as hk with $h \in P$, $k \in G(\mathfrak{o})$. We have

$$[\,0 \quad 0 \quad 1\,]\,\bar{n} = [\,z \quad y \quad 1\,]\,.$$

which if $z \notin \mathfrak{o}$ we can normalize to

$$[\,1 \quad y/z \quad 1/z\,]\,.$$

We finally find

$$\bar{n} = \begin{bmatrix} 1/\overline{z} & y/z & 1 \\ 0 & \overline{z}/z & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ x/z & 1 & 0 \\ 1/z & y/z & 0 \end{bmatrix} w\,.$$

We filter N by subgroups N_n where z lies in $\mathfrak{p}_{E_\bullet}^n$. The quotient N_0/N_1 has size q_E^3, the quotient N_1/N_2 has size q_E. The groups N_{even} are all conjugate, as are the groups N_{odd}.

Let χ be the character

$$\begin{bmatrix} y & 0 & 0 \\ 0 & \overline{y}/y & 0 \\ 0 & 0 & 1/\overline{y} \end{bmatrix} \mapsto |y|_{E_\bullet}^s\,.$$

By expressing the integral for τ over $\overline{N}$ as the sum of integrals over N_0, $N_{-1} - N_0$, etc. we find that

$$\tau\varphi_s = c(s)\varphi_{-s}$$

with

$$c(s) = \frac{\big(1 - q_E^{-2}\chi(\alpha^\vee(\varpi))\big)\big(1 + q_E^{-1}\chi(\alpha^\vee(\varpi))\big)}{1 - \chi(\alpha^\vee(\varpi))^2} \,.$$

For the calculation, let $X = (\chi\delta_P^{1/2}(\alpha^\vee(\varpi))$. Then the integral is

$$\begin{aligned} 1 + (q-1)X + (q^4 - q)X^2 + (q^5 - q^4)X^3 + (q^8 - q^5)X^4 + \cdots \\ &= 1 + \big((q-1)X + (q^4 - q)X^2\big)\big(1 + q^4X^2 + q^8X^4 + \cdots\ \big) \\ &= \frac{(1 - q^4X^2) + (qX - X) + (q^4X^2 - qX^2)}{1 - q^4X^2} \\ &= \frac{(1 - X)(1 + qX)}{1 - q^4X^2} \,. \end{aligned}$$

Now let's interpret this in terms of the L-group, which I have already partly described in an earlier section. The L-group ${}^LG_{E_\bullet/E}$ is the semi-direct product of $PGL_3(\mathbb{C})$ and $\{1, \sigma\}$, and the L-group $\widehat{G}_{E_\bullet/F}$ is the product of several copies of this and an induced action of the cyclic Galois group of E/F. Again let $\hat{\mathfrak{n}}^{\mathrm{opp}}$ be the negative root space in $\widehat{G}$. I now claim that

$$c(s) = \frac{\det_{\hat{\mathfrak{n}}^{\mathrm{opp}}}\big(I - (\hat{t}_\chi \times \mathfrak{Frob})\big)^{-1}}{\det_{\hat{\mathfrak{n}}^{\mathrm{opp}}}\big(I - q_F^{-1}(\hat{t}_\chi \times \mathfrak{Frob})\big)^{-1}} \,.$$

To see this, we just have to calculate $\mathrm{Ad}_{\hat{\mathfrak{n}}^{\mathrm{opp}}}(\hat{t}_\chi \times \mathfrak{Frob})$. Here $\hat{t}$ is chosen so

$$t_1/t_3 = \chi(\alpha^\vee(\varpi^{-1})), \quad t_3/t_1 = \chi(\alpha^\vee(\varpi)) \,.$$

But we can calculate

$$\begin{aligned} \mathfrak{Frob}\colon\quad & \bar{x}_{1,2} \mapsto \bar{x}_{2,3} \\ & \bar{x}_{2,3} \mapsto \bar{x}_{1,2} \\ & \bar{x}_{1,3} \mapsto -\bar{x}_{1,3} \\ \hat{t} \times \mathfrak{Frob}\colon\quad & \bar{x}_{1,2} \mapsto (t_3/t_2)\bar{x}_{2,3} \\ & \bar{x}_{2,3} \mapsto (t_2/t_1)\bar{x}_{1,2} \\ & \bar{x}_{1,3} \mapsto -(t_3/t_1)\bar{x}_{1,3} \end{aligned}$$

so that its matrix is

$$\begin{bmatrix} 0 & t_2/t_1 & 0 \\ t_3/t_2 & 0 & 0 \\ 0 & 0 & -(t_3/t_1) \end{bmatrix}$$

from which the claim can be verified.

- *Representations induced from a Borel subgroup*

Let now G be an arbitrary unramified reductive group defined over F, let B be a Borel subgroup, T a maximal torus in B, W the corresponding Weyl group. For each unramified character χ of T and Borel subgroups P and Q containing T let

$$\tau = \tau_{Q,P,\chi}\colon\ \mathrm{Ind}\big(\chi \mid P(F), G(F)\big) \longrightarrow \mathrm{Ind}\big(\chi \mid Q(F), G(F)\big)$$

be the intertwining operator defined formally by

$$\tau\varphi(g) = \int_{N_Q(F)\cap N_P(F)\backslash N_Q(F)} \varphi(ng)\, dn\,.$$

If x, y are elements of W with $\ell(xy) = \ell(x) + \ell(y)$, then we have a kind of functional equation

$$\tau_{xyBy^{-1}x^{-1},B} = \tau_{xyBy^{-1}x^{-1},xBx^{-1}}\tau_{xBx^{-1},B}\,.$$

For any unramified character χ and Borel subgroup P containing T there exists a unique function $\varphi_{\chi,P}$ in $\mathrm{Ind}\big(\chi \mid P(F), G(F)\big)$ fixed by K with $\varphi_{\chi,P}(1) = 1$. From the functional equation just above and the rank one calculations made earlier we can deduce easily that

$$\tau_{wBw^{-1},B}\varphi_{\chi,B} = c(\chi)\varphi_{\chi,wBw^{-1}}$$

and

$$c(\chi) = \frac{\det_{\hat{\mathfrak{n}}_w^{\mathrm{opp}}}\big(I - (\hat{t}_\chi \times \mathfrak{Frob})\big)^{-1}}{\det_{\hat{\mathfrak{n}}_w^{\mathrm{opp}}}\big(I - q_F^{-1}(\hat{t}_\chi \times \mathfrak{Frob})\big)^{-1}}$$

where

$$\mathfrak{n}_w = \hat{\mathfrak{n}}^{\mathrm{opp}}/w\hat{\mathfrak{n}}^{\mathrm{opp}}w^{-1} \cap \hat{\mathfrak{n}}^{\mathrm{opp}}\,.$$

- *Representations induced from opposite parabolic subgroups*

Suppose now that P is a parabolic subgroup of G, $\overline{P}$ an opposite, $M = P \cap \overline{P}$. If (σ, U) is an unramified representation of $M(F)$, then by the Satake isomorphism it corresponds to a conjugacy class $\hat{t}_\sigma \times \mathfrak{Frob}$ in the L-group of M. The same element represents in ${}^LG$ the unramified representation $\mathrm{Ind}\big(\sigma \mid P(F), G(F)\big)$ of $G(F)$. Suppose given in U a particular vector φ_U fixed by $M(\mathfrak{o})$. In the induced representation $\mathrm{Ind}\big(\sigma \mid M(F), G(F)\big)$ there will be a unique function φ_σ fixed by $G(\mathfrak{o})$ and such that $\varphi_\sigma(1) = \varphi_U$. Define $\overline{\varphi}_\sigma$ similarly in the representation induced from $\overline{P}$.

Theorem. *In these circumstances we have*

$$\tau\varphi_\pi = c(\pi)\overline{\varphi}_\pi$$

where

$$c(\pi) = \frac{\det_{\hat{\mathfrak{n}}^{\mathrm{opp}}}\big(I - (\hat{t}_\pi \times \mathfrak{Frob})\big)^{-1}}{\det_{\hat{\mathfrak{n}}^{\mathrm{opp}}}\big(I - q_F^{-1}(\hat{t}_\pi \times \mathfrak{Frob})\big)^{-1}}$$

and $\hat{\mathfrak{n}}$ is the radical of $\widehat{P}$ in $\widehat{G}$.

The proof of this formula follows almost immediately from the one in the previous section, because induction from parabolic subgroups is transitive.

- *The global constant term*

Now we consider things globally. Let P be a maximal proper rational parabolic subgroup of the rational group G, σ a cuspidal automorphic representation of $M(\mathbb{A})$, $\pi = \mathrm{Ind}\big(\sigma \mid P(F), G(F)\big)$. Let $\hat{t}_\delta$ be the element of $\widehat{T}$ representing the modulus character δ_P. It lies in fact in the center of ${}^L M$, since δ_P is a character of M. It is also the image in $\widehat{T}$ of the product

$$\prod_{\alpha \in \Sigma_P^+} \alpha^\vee(\varpi^{-1}) \, .$$

The vector space $\hat{\mathfrak{n}}^{\mathrm{opp}}$ decomposes under $\hat{t}_\delta$ into eigenspaces with eigenvalues a_i. Let ρ_i be the representation of ${}^L M$ on the eigenspace for a_i.

The constant term of the Eisenstein series corresponding to the local functions φ_s

$$\prod_1^r \frac{L(a_i s, \rho_i, \pi)}{L(a_i s + 1, \rho_i, \pi)} \, .$$

In favourable cases (for example, when $r = 1$) this implies that the L-function has a meromorphic continuation. More about exactly which L-functions arise is discussed in some detail in the *Euler Products* notes. I should add that although it was certainly impressive that Langlands was able to use the theory of Eisenstein series to prove in one stroke that several new families of L-functions possessed a meromorphic continuation, the technique was certainly limited. As observed by Langlands himself, perhaps the most striking case was that where $G = G_2$ and $M = GL_2$.

A similar calculation for other terms in the Fourier expansion of Eisenstein series, suggested by Langlands in the 1967 letter to Godement and carried out in detail much later by Shahidi, derives a functional equation for the L-function in the cases where σ has a Whittaker model.

Langlands tells me that the L-functions arising in the constant term of Eisenstein series played a crucial role in his thinking, but exactly what role is not clear to me. In the notes on Euler products he credits Jacques Tits with the observation that they are of the form $L(s, \rho, \pi)$ where ρ is the representation on the nilpotent Lie algebra, but as far as I can see Tits could only have made this observation in Langlands' lectures at Yale in May, 1967, several months after the letter to Weil.

11. Some subsequent developments

Langlands realized the importance of the L-group much more clearly than any to whom he explained his conjectures. He immediately began to work out various ways in which it played a role. Already in May, 1967, we find him writing a letter to Godement conjecturing a formula relating Whittaker functions to the Weyl character formula applied to the L-group. (This was later to become the formula of Casselman and Shalika, who learned only after they had proven it that Langlands had conjectured it seven years before!)

Questions raised by his conjectures presumably motivated his exhaustive investigation of local L-factors and root numbers, later simplified to some extent by Deligne. The local functoriality principle received striking evidence from his work on the ℓ-adic representations of modular varieties for presentation at the 1972 Antwerp conference, which I have already alluded to.

But perhaps most interesting was the appearance of phenomena related to L-**indistinguishability**. We have already seen that to some extent the functoriality principle asserted a kind of characterization of an automorphic form, or equivalently an irreducible representation of $G(\mathbb{A})$, by its Frobenius classes. But what happens for GL_2 turns out to be deceiving. For other groups, representations both global and local come in equivalence classes called L-indistinguishable, which means that as far as their L-functions are concerned they appear to be the same. For GL_n, each equivalence class has just a single element in it. This notion of equivalence among representations turned out to be related to a simple equivalence relation on conjugacy classes. Both these notions turned out to be necessary to understand the exact relationship between the trace formula and the Hasse-Weil zeta functions of Shimura varieties. Questions raised in this way were surprisingly subtle and complicated, and have occupied many first-rate mathematicians since they were brought to public attention in Langlands' lectures at the University of Paris. Many of the most difficult, but presumably not impossibly difficult, open questions in the subject are concerned with these issues. (A succinct and admirable discussion of these matters was presented by Arthur at the Edinburgh conference.)

Another extremely interesting development was the extension of local functoriality to include Galois representations with a large unipotent component, for example those arising from elliptic curves with multiplicative reduction. Here arose the **Deligne-Langlands conjecture**, which predicted a complete classification of square-integrable representations of $\mathfrak{p}$-adic reductive groups occurring as subrepresentations of the unramified principal series. This conjecture was eventually proven by Kazhdan and Lusztig. Related matters were investigated in a long series of papers by Lusztig on the Hecke algebras associated to affine Weyl groups, where perhaps for the first time the L-group occurred as a geometrical object. In particular, L-indistinguishability appeared naturally in terms of local systems on the L-group.

12. Things to look for

One can find elsewhere accounts of serious and outrageously difficult conjectures implicit in Langlands' construction of the L-group and Arthur's generalization of functoriality. I will not recall these conjectures, but instead I will pose here a number of more frivolous questions which are presumably more easily answered.

- Even in the case of compact quotients, the role of L-indistinguishability in the Arthur-Selberg trace formula is not at all clear, as Arthur points out in his Edinburgh exposé. This is presumably related to the rather formal aspect of proofs of the trace formula. Can one use ideas of Patterson, Bunke, and Olberich to elucidate the nature of L-indistinguishability?
- Even more formal are Arthur's arguments for non-compact quotients. What sort of analysis or geometry would make the trace formula seem natural? This is somewhat mysterious even for $SL_2(\mathbb{Q})$.
- Recently, following an extraordinary paper of Lusztig where intersection cohomology and the Weyl character formula appear together, Ginzburg and others have formulated the Satake isomorphism in terms of tensor categories of sheaves on a kind of Grassmannian associated to a group over fields of the form $F((t))$. In this context, the L-group is defined for the first time as a group rather than just formally. Is there a version of this valid for $\mathfrak{p}$-adic groups? Can one formulate and prove classical local class field theory in these terms? It is difficult to believe that one will ever understand the conjectured relationship between local Galois groups

and representations of $\mathfrak{p}$-adic groups until one has a formulation of local class field theory along these lines.

- Geometry of the L-group first appeared, as I have already mentioned, in Lusztig's work on the conjecture of Deligne-Langlands. Lusztig showed in this work that Kazhdan-Lusztig cells in affine Weyl groups were strongly related to unipotent conjugacy classes in the L-group. Apparently still unproven remain conjectures of Lusztig relating such cells to cohomology of subvarieties in the flag manifold of the L-group.
- What replaces the L-group in analyzing Kazhdan-Lusztig cells in hyperbolic Coxeter groups? The phenomena to be explained can be found in work of Robert Bédard, but not even the merest hint of what to do with them.
- Manin tells us that we should think of algebraic varieties at real primes as having the worst possible reduction. Is there any way one can use this idea to make better sense of representations of real groups? Can we use representation theory of either real or $\mathfrak{p}$-adic groups to explain Manin's formulas in Arakelov geometry?
- In his Zürich talk, Rapoport mentioned a possible approach to local functoriality conjectured by Kottwitz and Drinfeld. The idea is highly conjectural, but any progress here would be interesting.
- Another approach to local functoriality was mentioned in Ginzburg's talk at the ICM in Berkeley. This looks more interesting in light of the 'new' Satake isomorphism. Is there anything to it?
- One of the the oddest puzzles in the theory of local L-functions in representation theory is the necessity of introducing an additive character to define the ϵ-factors. There are two places in local representation theory where these arise naturally—in the theory of Godement-Jacquet for GL_n and in the theory of Whittaker functions, which play a puzzling role. In a recent paper, Frenkel et al. interpret the explicit formula of Casselman-Shalika for Whittaker functions in geometric terms. It would be interesting—illuminating both the meaning of local L-functions and the L-group—if one could prove the formula in this context. It would also be interesting if one could similarly understand Mark Reeder's generalization of the Casselman-Shalika formula.
- I have proven above a formula for the effect of interwining operators on unramified functions on a $\mathfrak{p}$-adic group, which has a striking formulation in terms of the L-group. The proof is entirely computational, however. Can one explain this formula directly in terms of the L-group? Extend it to ramified representations? Similarly deduce Macdonald's formula for unramified matrix coefficients?
- There has been a lot of work on the classification of irreducible representations of local reductive groups in the past several years, but the Galois group plays no apparent role in these investigations. Is there any way to introduce it there? Is there any way to generalize Kazhdan-Lusztig's work on the Deligne-Langlands conjecture to ramified representations?

I refrain from commenting on overlap among these problems.

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